RELATIVISTIC QUANTUM MECHANICS

and

INTRODUCTION TO QUANTUM FIELD THEORY

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References

This very short list of references points out some textbooks or lecture notes of easy reading.

Textbooks

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Contents

Foreword

This introductory course addresses some aspects of relativistic quantum mechanics, provides the basic principles of Quantum Field Theory (QFT) and should contribute to an easy reading of general textbooks on the subject. It develops a good understanding of the key ideas of field theory and introduces some calculation methods, but is surely not complete in the formulation of perturbative expansions and in the discussion of the Feynman diagram techniques. QFT is the theory which best describes elementary particles and their interactions. It is automatically a many-particle theory and allows to perform calculations that have accurate agreements with experiment. For example, in Quantum Electrodynamics (QED), the anomalous electron magnetic moment and the Lamb shift (splitting between $2s_{1/2}$ and $2p_{1/2}$ states of H-atom) are predicted with high precision. However, field theory is not entirely satisfactory and cannot be compared with the nice mathematical theory of general relativity. When we try to use it to calculate physical quantities, we encounter infinite results. Making sense of these infinities, *i.e.* performing renormalization, occupies a large part of any books of QFT. This question will unfortunately not be addressed in this lecture.

A field is a mathematical quantity which takes a value at every point in space-time. We are already familiar with electric and magnetic vector fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$. The real or complex scalar field $\phi(\mathbf{r},t)$ is another example. Generally speaking, fields are classified according to their behaviour under symmetry transformations. All along this lecture, we will first consider the field as a classical quantity whose values, at each space-time position, play the role of dynamical coordinates.Then, the quantization will be performed by direct application of the canonical quantization rules. Even if QFT is our announced goal, an important part of this course will be devoted to relativistic quantum mechanics. Generally, relativistic quantum mechanics and QFT are studied separately. Here, we may try to present them together by continually emphasizing their differences. The main difference between the two essentially concerns the number of particles. Both are using tensor quantities like contravariant or covariant space-time four-vectors

$$(x^\mu) = (ct, \mathbf{r}) \qquad (x_\mu) = (ct, -\mathbf{r}) \qquad \mu = 0,1,2,3 \tag{1}$$

related by the metric

$$[g^{\mu\nu}] = [g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad x_\mu = g_{\mu\alpha}x^\alpha. \tag{2}$$

The transition from one to the other Lorentzian frame is given by the linear transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{3}$$

where $(\Lambda^\mu{}_\nu) = \Lambda \in \{\Lambda / \Lambda^T g \Lambda = g\}$ is a Lorentz matrix. The derivatives are given by the contravariant and covariant components

$$(\partial^\mu) \equiv (\frac{\partial}{\partial x_\mu}) = (\partial_{ct}, -\nabla) \qquad\qquad (\partial_\mu) \equiv (\frac{\partial}{\partial x^\mu}) = (\partial_{ct}, \nabla) \tag{4}$$

and it must be recalled that the components of the derivative with respect to x_μ are contravariant, but with a minus sign in the last three components.

In this lecture, QFT will be expressed in **natural units**

$$\hbar = c = \epsilon_0 = 1 \ . \tag{5}$$

Then, by considering the well-known energy formulas

$$\begin{array}{ll} E = mc^2 & \text{Energy and mass equivalence} \\ E = \dfrac{hc}{\lambda} & \text{De Broglie wavelength} \\ E = h\nu & \text{Einstein photon energy ,} \end{array}$$

we see that in natural units mass M, length L and time T have the dimension of a power of the energy

$$[M] = eV \qquad [L] = eV^{-1} \qquad [T] = eV^{-1}. \tag{6}$$

A physical quantity Q depending on M, L, T can be converted from natural units to SI units by dimensional analysis

$$[Q]_{SI} = [Q]_{NU} \ [\hbar]^\mu \ [c]^\nu \ . \tag{7}$$

This equation allows to determine the exponents μ and ν. As an example, we consider the energy density W given by the formula

$$W(\mathbf{r}) = \int \frac{d^3p}{(2\pi)^3} \ \sqrt{m^2 + \mathbf{p}^2} \ e^{i\mathbf{p}\cdot\mathbf{r}} \tag{8}$$

where $\mathbf{p}$ is the vector momentum and where the argument of the exponential function must obviously be dimensionless. A quick look at formula (8) shows that, in SI units, we have the energy factor $\epsilon_{\mathbf{p}} = \sqrt{(mc^2)^2 + (\mathbf{p}c)^2}$ and the dimensionless argument $\mathbf{p} \cdot \mathbf{r}/\hbar$. Then, Eq. (7) and Eq. (8) allow to write the dimensional equation

$$(ML^2T^{-2})L^{-3} = (MLT^{-1})^3(ML^2T^{-2}) \ \ (ML^2T^{-1})^\mu(LT^{-1})^\nu, \tag{9}$$

from where we deduce $\mu = -3$, $\nu = 0$ and finally get the formula (8) in SI units

$$W = \int \frac{d^3p}{(2\pi\hbar)^3} \ \sqrt{(mc^2)^2 + (\mathbf{p}c)^2} \ \ e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \ . \tag{10}$$

This dimensional analysis is valid for kinematic units. For electric units, the electric charge must be considered and replaced by the dimensionless fine structure constant α

$$\frac{e^2}{4\pi} \quad \longrightarrow \quad \alpha = \frac{e^2}{4\pi\epsilon_0} \ \frac{1}{\hbar c} \ . \tag{11}$$

Chapter 1

Introduction : From Classical Mechanics to Field Theory

The term classical field is somehow misleading since most of the fields we will encounter in this course arise from quantum mechanics. Independently of their quantum origin, they are first treated within the framework of classical mechanics. They become quantum fields, as soon as they are considered as operators acting on a Hilbert space and subject to canonical quantization rules. This process received the inappropriate name of **second quantization**.

One can grasp the proper meaning of field quantization by taking a simple example. A function $\psi(\mathbf{r}, t)$ can generally be seen as a classical field governed by some Schrödinger-like equation. On one side, if ψ is treated as probability amplitude *i.e.* as an element of the Hilbert space of square integrable functions, we are involved in the quantum mechanics of a one-particle system. On the other side, if the field ψ is interpreted as an operator acting on a many-particle Hilbert space or Fock space, we are involved in the QFT. Actually, there is only one quantization, but seen from different point of views. Both approaches make sense as soon as the link between a many-particle system and a field theory is understood. Actually, depending on the physical system we are considering, the function $\psi(\mathbf{r}, t)$ can have three accepted meanings : classical field, probability amplitude or quantum field.

The best-known classical field we should primarily consider is the electromagnetic field, which does not possess a one-particle interpretation. Its classical treatment has already been discussed in the basic course of electrodynamics and its quantum interpretation as photons of energy $E = \hbar\nu$ appears in the introduction of quantum mechanics. However, the genuine quantization of the electromagnetic field requires methods of quantum field theory and brings out some difficulties which are specific to this massless field. We will deal with at some later time.

This chapter introduces the main ideas of field theory. In a first step, it considers the link between a N-particle system and a field by discussing the transition from a chain of harmonic oscillators to a vibrating string. In a second step, it addresses the quantization of the field in the light of the quantized simple harmonic oscillator[1]. A good understanding of the concepts developed in this chapter is necessary for grasping the essential features of QFT.

[1]The single-particle quantum harmonic oscillator is thoroughly treated in the basic course of quantum mechanics. Harmonic oscillator is very common in physics as Sidney Coleman says : "*The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.*"

1.1 Mechanics and Quantum Mechanics of a System of Particles

A mechanical system of n degrees of freedom can be characterized by a Lagrangian

$$L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t) \tag{1.1}$$

depending on generalized coordinates q_j and generalized velocities $\dot{q}_j$. The application of the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \; dt = 0 \tag{1.2}$$

leads to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, \cdots, n \; . \tag{1.3}$$

From the definition of the conjugate momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{1.4}$$

and by using a Legendre transformation we get the Hamiltonian

$$H(q_1, \cdots, q_n, p_1, \cdots, p_n, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t) \; . \tag{1.5}$$

The calculation of the total differential of each member of this relation allows to derive the canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad\qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{1.6}$$

The Poisson's bracket defined as

$$\{f, g\} = \sum_{j=1}^{n} \left[\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} \right] \tag{1.7}$$

leads to the canonical relations

$$\{p_j, q_k\} = \delta_{jk} \tag{1.8}$$

$$\{q_j, q_k\} = 0 = \{p_j, p_k\} \; . \tag{1.9}$$

The quantization of this system of n degrees of freedom proceeds by replacing the canonical variables q_j, p_j by time-dependent operators[2] acting on a Hilbert space and obeying the canonical **commutation relations**[3]

$$\Big[q_j(t), p_k(t)\Big] = i\hbar \, \delta_{jk} \tag{1.10}$$

$$\Big[q_j(t), q_k(t)\Big] = 0 = \Big[p_j(t), p_k(t)\Big] \; . \tag{1.11}$$

[2]The operators $O(t)$ are written in Heisenberg representation $O(t) = e^{iHt} O e^{-iHt}$ where H is the Hamiltonian of the system.

[3]The commutator of two operators is defined as $[A, B] = AB - BA$.

1.2 Limit to the Continuum

The transition from an N-particle system to a classical field is illustrated by considering a one-dimensional chain of elastically coupled atoms of mass m and coupling strength K. At equilibrium, the atoms are at positions ia, $i = 1, \cdots, N$ as depicted in Figure 1.1. The motion of the atoms around these equilibrium points is described by the displacement[4] coordinates $\phi_i(t)$. It is governed by the Lagrangian containing a kinetic term T and an elastic potential term V

$$L = T - V = \sum_{i=1}^{N} \left[\frac{1}{2} m \dot{\phi}_i^2 - \frac{1}{2} K \left(\phi_{i+1} - \phi_i \right)^2 \right]. \tag{1.12}$$

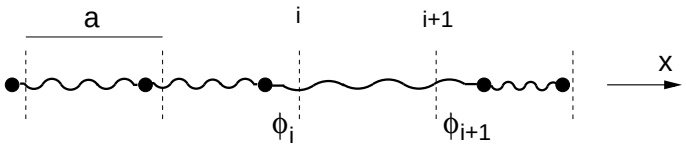

Figure 1.1: Chain of elastically coupled atoms

We assume periodic boundary conditions $\phi_{N+1}(t) = \phi_1(t)$. This system of N particles could be treated by methods of classical mechanics. We are rather interested to the continuum limit $a \to 0$, for a fixed chain length $\ell = Na$. Thus the number of degrees of freedom diverges. Introducing the parameters $x_i = ia$, the mass density $\mu = m/a$, the elastic modulus $Y = Ka$ and the displacement $\phi(x_i, t) = \phi_i(t)$ as a function of the equilibrium position of the atoms, the Lagrangian (1.12) takes the form

$$\begin{aligned} L &= a \sum_{i=1}^{N} \left[\frac{1}{2} \frac{m}{a} \dot{\phi}_i(t)^2 - \frac{1}{2} Ka \left(\frac{\phi_{i+1}(t) - \phi_i(t)}{a} \right)^2 \right] \\ &= a \sum_{i=1}^{N} \left[\frac{1}{2} \mu \left(\frac{\partial \phi(x_i, t)}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\phi(x_i + a, t) - \phi(x_i, t)}{a} \right)^2 \right] \\ &= a \sum_{i=1}^{N} L_i \ . \end{aligned} \tag{1.13}$$

At the limit $a \to 0$, the above Riemann sum is tranformed into the integral

$$\begin{aligned} L &= \int_0^\ell dx \left[\frac{1}{2} \mu \left(\frac{\partial \phi(x, t)}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial \phi(x, t)}{\partial x} \right)^2 \right] \\ &= \int_0^\ell dx \ \mathcal{L}(\partial_t \phi, \partial_x \phi) \end{aligned} \tag{1.14}$$

where we have defined the **Lagrangian density** $\mathcal{L}$. The Lagrangian L describes the longitudinal vibrations in a continuum medium or along a vibrating string. The continuous

[4]The positions of the masses m on the x-axis are given by $x_i(t) = \bar{x}_i + \phi_i(t)$ where $\bar{x}_i = ia$.

parameter x is not a dynamical variable, it serves merely as a continuous index replacing the index i. Between discrete and continuous sytems, exists the following correspondence

$$\begin{array}{lcl} \phi_i(t) & \longrightarrow & \phi(x,t) \\ i = 1, \cdots, N \ , \qquad L = a \sum_i L_i & & x \in [0, \ell] \ , \qquad L = \int \mathcal{L} \ dx \ . \end{array} \tag{1.15}$$

The function $\phi(x,t)$ is called **scalar field** and the dynamical coordinates are the values of this field at every space point. Thus, a field has an uncountably infinite number of degrees of freedom, which is the source of many of the difficulties in field theory. The sum over the label i is replaced by an integral over dx. The spatial derivatives arise naturally from the terms coupling neighbouring space points, when the separation goes to zero. The classical mechanical treatment of this one-dimensional field can be carried on with the equation of motion following from the Hamilton's principle

$$\delta S \equiv \delta \int_{t_1}^{t_2} dt \int_0^{\ell} dx \ \mathcal{L} = 0 \tag{1.16}$$

subject to the condition $\delta\phi = 0$ at the boundary points. The variation (1.16) is calculated in the usual way and gives

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} dt \int_0^{\ell} dx \left[\frac{\mu}{2} (\partial_t \phi)^2 - \frac{Y}{2} (\partial_x \phi)^2 \right] \\ &= \int_{t_1}^{t_2} dt \int_0^{\ell} dx \Big[\mu \ \partial_t \phi \ \delta(\partial_t \phi) - Y \ \partial_x \phi \ \delta(\partial_x \phi) \Big] \ . \end{aligned}$$

By using the commutativity property of derivative and variation and by performing an integration by parts we obtain

$$\int_{t_1}^{t_2} dt \int_0^{\ell} dx \left[\mu \ \partial_t^2 \phi \ - Y \ \partial_x^2 \phi \right] \delta\phi - \mu \int_0^{\ell} dx \ \partial_t \phi \ \delta\phi \Big|_{t_1}^{t_2} + Y \int_{t_1}^{t_2} dt \ \partial_x \phi \ \delta\phi \Big|_0^{\ell} = 0 \ .$$

Finally, by using boundary conditions and by applying the fundamental lemma of variational calculus, we get the **field equation**

$$\left[\partial_t^2 - v^2 \partial_x^2 \right] \phi(x,t) = 0 \tag{1.17}$$

which is nothing else then the wave equation for a vibrating string, where $v^2 = Y/\mu$. The next step towards Hamiltonian mechanics requires new mathematical tools, since the Lagrangian L (1.14) is becoming a **functional**[5] and the conjugate momentum $\pi(x,t)$ must be defined as **functional derivative**[6] of L with respect to the velocity $\dot{\phi} = \partial_t \phi$

$$\pi = \frac{\delta L}{\delta \dot{\phi}} \ . \tag{1.18}$$

[5] A functional F is an application from the space of functions f into $\mathbb{R}$, *i.e.* $f \longmapsto \int \mathcal{F}(f(x)) \ dx$.

[6] The variation δF of a functional F is a linear map defined by the expression

$$F[f+h] - F[f] - \delta F[h] = \mathcal{O}(h) \qquad \lim_{\|h\| \to 0} \mathcal{O}(h)/\|h\| = 0 \ .$$

The functional derivative $\delta F / \delta f$ is defined by the integral

$$\delta F[h] = \int \frac{\delta F}{\delta f(x)} h(x) \ dx \ .$$

This definition becomes obvious if we consider the differential $dF[h] = \sum_{j=1}^{N} \partial F / \partial f_j \ h_j$ of a function of N variables and take the continuum limit $N \to \infty$.

However, from the integral form (1.14) of L and by using the definition of the functional derivative, we can see (do it !) that the conjugate momentum is equivalent to the partial derivative of the Lagrangian density

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \ . \tag{1.19}$$

This can also be seen by considering a discretized Lagrangian integral. From the Lagrangian density (1.14), we obtain the conjugate momentum

$$\pi = \mu \dot{\phi} \tag{1.20}$$

and deduce the Hamiltonian of a vibrating string

$$\begin{aligned} H &= \int_0^\ell dx \left[\pi \dot{\phi} - \mathcal{L} \right] \\ &= \int_0^\ell dx \left[\frac{\pi^2}{2\mu} + \frac{Y}{2} \left(\partial_x \phi \right)^2 \right] \ . \end{aligned} \tag{1.21}$$

The solution of the linear differential equation (1.17) must satisfy periodic boundary conditions in the interval $[0, \ell]$. It can be found by expanding ϕ in a Fourier series

$$\phi(x,t) = \frac{1}{\sqrt{\ell}} \sum_k \phi_k(t) e^{ikx} \tag{1.22}$$

where k takes the discrete values $k = 2\pi n/\ell$, $n \in \mathbb{Z}$. The Hilbert basis $e^{ikx}/\sqrt{\ell}$ has the orthonormality property

$$\frac{1}{\ell} \int_0^\ell e^{i(k-k')x} dx = \delta_{kk'} \tag{1.23}$$

and the complex Fourier coefficients are then given by the integral

$$\phi_k(t) = \frac{1}{\sqrt{\ell}} \int_0^\ell \phi(x,t) \ e^{-ikx} dx \ . \tag{1.24}$$

If we impose the real field condition $\phi(x,t) = \phi(x,t)^*$, we obtain

$$\phi_k(t) = \phi^*_{-k}(t) \ , \tag{1.25}$$

and it is easy to check (do it !) the Parseval's relation

$$\int_0^\ell \phi(x,t)^2 dx = \sum_k \phi_k(t) \phi_{-k}(t) \tag{1.26}$$

and the closure relation

$$\frac{1}{\ell} \sum_k e^{ik(x-x')} = \delta(x - x'). \tag{1.27}$$

The Fourier expansion (1.22) inserted into the wave equation (1.17) provides the differential equation

$$\frac{d^2 \phi_k}{dt^2} + \omega_k^2 \phi_k = 0 \tag{1.28}$$

where $\omega_k = v|k|$. The general solution of this second order linear differential equation is given by the linear combination

$$\phi_k = A_k e^{-i\omega_{\mathbf{k}} t} + B_k e^{i\omega_{\mathbf{k}} t} \qquad A_k, B_k \in \mathbb{C} \ . \tag{1.29}$$

Because of the choice of complex basis functions, the coefficients A_k, B_k must be complex, but the real field condition (1.25) imposes $B_k = A^*_{-k}$. Thus, the field (1.22) can be written as a superposition of normal modes

$$\begin{aligned} \phi(x,t) &= \frac{1}{\sqrt{\ell}} \sum_k [A_k \ e^{-i\omega_{\mathbf{k}} t} + A^*_{-k} \ e^{i\omega_{\mathbf{k}} t}] \ e^{ikx} \\ &= \frac{1}{\sqrt{\ell}} \sum_k [A_k \ e^{-i(\omega_{\mathbf{k}} t - kx)} + A^*_k \ e^{+i(\omega_{\mathbf{k}} t - kx)}] \ . \end{aligned} \tag{1.30}$$

With this Fourier expansion and by using Parseval's relation (1.26), the classical field Hamiltonian (1.21) can be brought (homework) into the form

$$H = \sum_k \omega_k^2 (A^*_k A_k + A_k A^*_k) \tag{1.31}$$

where we have set $\mu = 1$ by simply changing the units of the field ϕ.

1.3 Field Quantization and Fock Space

It is now possible to see the natural emergence of **field quantization**[7]. Indeed, the Hamiltonian structure (1.31), similar to a sum of quantum harmonic oscillators, suggests to consider the Fourier coefficients A_k, A^*_k as lowering and raising operators acting on a Hilbert space[8]. In particular, the complex conjugate number A^*_k is replaced by the **adjoint operator**[9]

$$A^*_k \quad \rightarrow \quad A^\dagger_k \ . \tag{1.32}$$

We reintroduce, for a while, the Planck's constant $\hbar$ in order to compare the Hamiltonian (1.31) with the known results from quantum mechanics. A change of normalization brings new operators.

$$a_k = \sqrt{\frac{2\omega_k}{\hbar}} \ A_k \tag{1.33}$$

and put the Hamiltonian (1.31) into the form

$$H = \frac{1}{2} \sum_k \hbar\omega_k \left(a^\dagger_k a_k + a_k a^\dagger_k \right) \tag{1.34}$$

[7]Here, it is not possible to directly use the quantization of the point mechanics. To quantize a continuum system, we need a specific procedure. The electromagnetic field presents the same problem.

[8]This Hilbert space, called Fock space, will be defined further. But it must not be confused with the Hilbert space of Fourier series that was mentioned above.

[9]In bracket notation, the adjoint $A^\dagger$ is defined as $\langle\varphi|A^\dagger|\psi\rangle = \langle\psi|A|\varphi\rangle^*$ and the symbol † is called "dagger". Moreover, the operators A_k stem from the coefficients of the Fourier expansion of the field, whereas in the single-particle harmonic oscillator they are defined from position and momentum operators.

which shows that a vibrating string can be described by an infinite number of non-interacting quantum harmonic oscillators, one oscillator for each eigenmode of the string motion. As for the single harmonic oscillator, commutation relations are assumed

$$\left[a_k, a_{k'}^\dagger)\right] = \delta_{kk'} \tag{1.35}$$

$$[a_k\ , a_{k'}] = 0 = \left[a_k^\dagger\ , a_{k'}^\dagger\right]. \tag{1.36}$$

Expressed as a function of the operators a_k and $a_k^\dagger$, the fields $\phi(x,t)$ and $\pi(x,t)$ become operators[10] too and read

$$\phi(x,t) = \frac{1}{\sqrt{\ell}} \sum_k \sqrt{\frac{\hbar}{2\omega_k}}\ [a_k\ e^{-i(\omega_k t - kx)} + a_k^\dagger\ e^{+i(\omega_k t - kx)}] \tag{1.37}$$

$$\pi(x,t) = \frac{1}{\sqrt{\ell}} \sum_k \sqrt{\frac{\hbar}{2\omega_k}}(i\omega_k)\ [-a_k\ e^{-i(\omega_k t - kx)} + a_k^\dagger\ e^{+i(\omega_k t - kx)}]. \tag{1.38}$$

From the commutations relations (1.35), (1.36) and by using the closure relation (1.27), we verify (homework) that the fields satisfy equal-time canonical **commutation relations**

$$[\phi(x,t), \pi(x',t)] = i\hbar\delta(x - x') \tag{1.39}$$

$$[\phi(x,t)\ , \phi(x',t)] = 0 = [\pi(x,t)\ , \pi(x',t)] \tag{1.40}$$

similar to those of the system of n degrees of freedom given in (1.11) and (1.10). Because of the continuous parameter x, the Kronecker symbol becomes a δ-function. Conversely, from (1.39), (1.40), the commutation relations of raising and lowering operators can be immediately recovered. The generalization to three dimension can be reached in a similar way[11]. The above field quantization approach will be repeated later for other fields like the Klein-Gordon field, the photon field and the fermion field. Only some technical aspects of the calculations will be different. Essentially, the sums will be replaced by integrals.

[10] These operators are not quantum observables, but they enter into the composition of observables such as charge, momentum or energy.

[11] The generalization to three dimensions is straightforward. We introduce orthogonal unit vectors $\mathbf{e}_n(\mathbf{k})$ for each $\mathbf{k}$ and for the three directions $n = 1, 2, 3$ in space. Then the vector expressions corresponding to (1.37) and (1.38) can be written

$$\mathbf{v}(\mathbf{r},t) = \frac{1}{\sqrt{\ell^3}} \sum_{\mathbf{k}} \sum_{n=1}^{3} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k},n}}}\ \mathbf{e}_n \left[a_{\mathbf{k},n} e^{-i(\omega_{\mathbf{k},n} t - \mathbf{k}\cdot\mathbf{r})} + a_{\mathbf{k},n}^\dagger\ e^{+i(\omega_{\mathbf{k},n} t - \mathbf{k}\cdot\mathbf{r})}\right],$$

$$\boldsymbol{\pi}(\mathbf{r},t) = \frac{1}{\sqrt{\ell^3}} \sum_{\mathbf{k}} \sum_{n=1}^{3} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k},n}}}\ \mathbf{e}_n \left[-a_{\mathbf{k},n} e^{-i(\omega_{\mathbf{k},n} t - \mathbf{k}\cdot\mathbf{r})} + a_{\mathbf{k},n}^\dagger\ e^{+i(\omega_{\mathbf{k},n} t - \mathbf{k}\cdot\mathbf{r})}\right] (i\omega_{\mathbf{k},n}).$$

The canonical commutation relations of the quantum fields are

$$[v_\alpha(\mathbf{r},t)\ , \pi_\alpha(\mathbf{r}',t)] = i\hbar\delta(\mathbf{r} - \mathbf{r}')\delta_{\alpha\beta}$$

$$[v_\alpha(\mathbf{r},t)\ , v_\beta(\mathbf{r}',t)] = 0 = [\pi_\alpha(\mathbf{r},t)\ , \pi_\beta(\mathbf{r}',t)]\ .$$

The Hilbert space upon which the field operators $a_k^\dagger\,, a_k$ are acting is constructed from the eigenstates of the single harmonic oscillator whose hamiltonian reads

$$H = \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right)\ . \tag{1.41}$$

These eigenstates are given by the expression

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad\qquad n = 0, 1, 2, 3, \cdots \tag{1.42}$$

where $|0\rangle$ is the one-particle ground state characterized by the property $a|0\rangle = 0$. We recall that the **raising** and **lowering** operators $a^\dagger$, a provide higher or lower states

$$a^\dagger|n\rangle = \sqrt{n+1}\ |n+1\rangle \qquad\qquad a|n\rangle = \sqrt{n}\ |n-1\rangle \tag{1.43}$$

and satisfy the commutation relations

$$\left[a, a^\dagger\right] = 1. \tag{1.44}$$

$$[a\ , a] = 0 = \left[a^\dagger\ , a^\dagger\right]\ . \tag{1.45}$$

The eigenvectors obey the orthonormalization condition

$$\langle n|n'\rangle = \delta_{nn'} \tag{1.46}$$

and the energy eigenvalue equation reads

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{1.47}$$

The field-operator Hamiltonian (1.34) consists of a sum of an infinite number of single harmonic oscillators. In order to identify the wave number k_j of each single oscillator, we write the Hamiltonian as a sum over j

$$H = \frac{1}{2}\sum_j \hbar\omega_{k_j}\left(a_{k_j}^\dagger a_{k_j} + a_{k_j}a_{k_j}^\dagger\right). \tag{1.48}$$

As usual, the eigenstates of the many-oscillator system are given by the tensor product of the eigenstates of the single oscillators. Then, the lowest energy state, called the **vacuum state**, takes the explicit form

$$|0\rangle \equiv |0_{k_1}, \cdots, 0_{k_j}, \cdots\rangle = |0\rangle_{k_1} \otimes \cdots \otimes |0\rangle_{k_j} \otimes \cdots \tag{1.49}$$

and will be annihilated by the operators a_{k_j}

$$I \otimes \cdots \otimes a_{k_j} \otimes I \otimes \cdots |0\rangle = 0 \qquad \text{for all} \quad k_j. \tag{1.50}$$

Higher energy states are given by the tensor products

$$|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle = |n_{k_1}\rangle \otimes \cdots \otimes |n_{k_j}\rangle \otimes \cdots \tag{1.51}$$

where each of the $n_{k_j} = 0, 1, 2, \cdots$ specifies the level of excitation of the k_j mode of the string. In a very natural way, these excitations of energy $\hbar\omega_{k_j}$ can be seen as **particles**[12], with n_{k_j} representing the number of particles in a given mode ω_{k_j}. In that case, the n_{k_j} are called **occupation numbers** and the Hilbert space spawned by the vectors $|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle$ is called **Fock space**. A Fock space is made from the direct sum of tensor products of single-particle Hilbert spaces

$$F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}. \tag{1.52}$$

The $a^\dagger_{k_j}$ become **creation** operators and the a_{k_j} **annihilation** operators. Their action on a vector of the Fock space reads

$$\begin{aligned} a^\dagger_{k_j}|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle &= \sqrt{n_{k_j}+1}\ |n_{k_1}, \cdots, n_{k_j}+1, \cdots\rangle & (1.53)\\ a_{k_j}|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle &= \sqrt{n_{k_j}}\ |n_{k_1}, \cdots, n_{k_j}-1, \cdots\rangle. & (1.54) \end{aligned}$$

The choice of the factors $\sqrt{n_{k_j}+1}$ and $\sqrt{n_{k_j}}$ preserves the normalization condition

$$\langle n_{k_1}, \cdots, n_{k_j} \cdots | n_{k'_1}, \cdots, n_{k'_j}, \cdots\rangle = \delta_{k_1 k'_1} \cdots \delta_{k_j k'_j} \cdots \tag{1.55}$$

and also the commutation relation[13]

$$\left[a_{k_j}\ ,\ a^\dagger_{k'_j}\right] = \delta_{k_j k'_j}. \tag{1.56}$$

The state $|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle$ is a state of $n = \sum_j n_{k_j}$ particles among which n_{k_j} are in the state of energy $\hbar\omega_{k_j}$. As it can be seen from (1.53) and (1.54), this state is an eigenstate of the **number operator**

$$N = \sum_j a^\dagger_{k_j} a_{k_j} \tag{1.57}$$

with eigenvalue n. Indeed, the calculations give

$$\begin{aligned} N|n_{k_1}, \cdots, n_{k_j}, \cdots\rangle &= \sum_j a^\dagger_{k_j} a_{k_j} |n_{k_1}, \cdots, n_{k_j}, \cdots\rangle \\ &= \sum_j a^\dagger_{k_j} \sqrt{n_{k_j}}\ |n_{k_1}, \cdots, n_{k_j}-1, \cdots\rangle \\ &= \sum_j \sqrt{n_{k_j}}\sqrt{n_{k_j}}\ |n_{k_1}, \cdots, n_{k_j}, \cdots\rangle \\ &= (n_{k_1} + n_{k_2} + \cdots)\ |n_{k_1}, \cdots, n_{k_j}, \cdots\rangle \\ &= n\ |n_{k_1}, \cdots, n_{k_j}, \cdots\rangle\ . \end{aligned} \tag{1.58}$$

[12]The oscillations of the string represent sound waves and the corresponding particles are called phonons. In the case of electromagnetic radiation, the particles are called photons. The above-defined Fock space is valid for bosons, where the occupation numbers n_{k_j} can take large values. Fock space for fermions will be defined later on.

[13]In the next chapters, fields and commutation relations will be subject to a Lorentz-invariant normalization condition.

More crucially, we remark that the ground state energy of the Hamiltonian (1.48) is infinite, since the action of H on the vacuum gives

$$\begin{aligned}\langle 0|H|0\rangle &= \frac{1}{2}\langle 0|\sum_j \hbar\omega_{k_j}\left(a^\dagger_{k_j}a_{k_j} + a_{k_j}a^\dagger_{k_j}\right)0\rangle \\ &= \langle 0|\sum_j \hbar\omega_{k_j}\left(a^\dagger_{k_j}a_{k_j} + \frac{1}{2}\right)|0\rangle \\ &= \sum_j \frac{1}{2}\hbar\omega_{k_j} \quad \longrightarrow \ \infty \ . \end{aligned} \tag{1.59}$$

However, we know that only energy differences can be observed. We therefore normalize the vacuum energy to 0 by convention and define the new ordered Hamiltonian

$$:H: \ = \sum_j \hbar\omega_{k_j} a^\dagger_{k_j}a_{k_j} \tag{1.60}$$

with the property

$$:H:|0\rangle = 0. \tag{1.61}$$

The symbol : : indicates that the creation operators are placed to the left of the annihilation operators in a so called **ordered product** or **normal product**

$$:a^\dagger_{k_j}a_{k_j} + a_{k_j}a^\dagger_{k_j}: \ = \ 2a^\dagger_{k_j}a_{k_j} \ . \tag{1.62}$$

The limit to the continuum nicely illustrates the two main steps of field quantization. On one side, it shows that the N-particle displacement coordinates become a field $\phi(x,t)$ whose Hamiltonian (1.31) looks like an infinite sum of single harmonic oscillators. On the other side, it explains how the field function $\phi(x,t)$ can be replaced by an operator obeying quantum commutation relations. In summary, field quantization can be characterized by the following two statements :

- classical field $\phi(x,t)$ plays the role of continuous dynamical variables indexed by x,
- field quantization considers the field $\phi(x,t)$ as an operator subject to canonical commutation relations and acting on a Fock space.

It is finally important to point out that quantum field theory introduces field operators satisfying canonical commutation relations and defines the corresponding quantum states. Many-body quantum theory uses the inverse approach. It defines many-particle symmetrized (antisymmetrized) quantum states on which creation and annihilation operators act and deduces the commutation (anticommutation) relations.

Chapter 2

Classical Field Mechanics

The concept of field mechanics has been introduced by considering the limit from a discrete system to the continuum. In this chapter, we give a very sketchy discussion of the classical field mechanics by considering a scalar field. The classical mechanics of other fields such as vector fields or spinor fields will be presented briefly later at the same time as their quantization. Additional informations on field classification, on Lagrangian expressions, on symmetries, etc. can be found in textbooks.

2.1 Lagrangian and Field Equations

The limit to the continuum discussed in the preceding chapter shows in particular the existence of a **Lagrangian density**

$$\mathcal{L}(\phi, \partial_\mu \phi) \tag{2.1}$$

depending on the field $\phi(x)$ and its first derivatives. The function argument written x represents the space-time coordinates $(t, \mathbf{r})$. The **Lagrangian** is given by spatial integration

$$L[\phi] = \int d^3r \ \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.2}$$

and therefore becomes a functional $L : \{\phi\} \longrightarrow \mathbb{R}$. Classical mechanics can be easily worked out. With the Lagrangian (2.2), we define the **action**

$$S = \int dt \ d^3r \ \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.3}$$

which is, because of the $d^4x \equiv dtd^3r$ integration, Lorentz-invariant. Then, the Hamilton's principle given by the variation condition

$$\delta S = 0 \tag{2.4}$$

yields (homework) the **Euler-Lagrange equation**[1]

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{2.5}$$

[1]The Einstein summation convention is understood on repeated indices $\mu = 1, 2, 3, 4$.

where as usual the condition $\delta\phi|_{\partial\Omega} = 0$ is assumed at the space-time boundary $\partial\Omega$. The extension to a many-component field ϕ_j is obvious. In order to establish the Hamiltonian mechanics, we define, as in (1.19), the **conjugate field** by the partial derivative of the Lagrangian density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \ . \tag{2.6}$$

It is now straightforward to write the **field Hamiltonian**

$$\begin{aligned} H &= \int d^3r \ \mathcal{H} \\ &= \int d^3r \left[\pi\dot{\phi} - \mathcal{L}\right] \end{aligned} \tag{2.7}$$

and to derive (homework) the **canonical equations** of motion

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \tag{2.8}$$

$$\dot{\pi} = -\frac{\delta H}{\delta \phi} \ . \tag{2.9}$$

Finally, for two functionals F and G, the **Poisson's bracket**

$$\{F, G\} = \int d^3r \left[\frac{\delta F}{\delta \pi}\frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \pi}\right] \tag{2.10}$$

provides (do it !) the classical **canonical relations**

$$\{\pi(\mathbf{r}, t), \phi(\mathbf{r}', t)\} = \delta(\mathbf{r} - \mathbf{r}') \tag{2.11}$$

$$\{\phi(\mathbf{r}, t), \phi(\mathbf{r}', t)\} = 0 \qquad \{\pi(\mathbf{r}, t), \pi(\mathbf{r}', t)\} = 0 \tag{2.12}$$

where the δ function replaces the Kronecker symbol appearing in the N-particle system.

2.2 Noether's Theorem

As in classical mechanics, the link between symmetry transformations and conserved quantities is provided by the **Noether's theorem**. However, in field theory the problem is a bit more complicated since both coordinates and fields are transformed. Symmetries are described by parameter-dependent transformations

$$x'^{\mu} = f^{\mu}(x, \alpha) \tag{2.13}$$

$$\phi'(x') = F(\phi(x), \alpha) \tag{2.14}$$

satisfying the conditions $f^{\mu}(x, 0) = x^{\mu}$ and $F(\phi, 0) = \phi$. The quantity α can represent several parameters. In order to study the local effect of these transformations, we write the relations (2.13) and (2.14) as **infinitesimal transformations**

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \delta x^{\mu} \\ &= x^{\mu} + f_i^{\mu}(x)\alpha_i + \mathcal{O}(\alpha^2) \end{aligned} \tag{2.15}$$

$$\begin{aligned} \phi'(x') &= \phi(x) + \delta\phi(x) \\ &= \phi(x) + F_i(x)\alpha_i + \mathcal{O}(\alpha^2) \end{aligned} \tag{2.16}$$

where δx^μ and $\delta\phi$ are first variations and the summation is implicit over the index i which enumerates the various parameters α_i. The functions $f_i^\mu(x)$ and $F_i(x)$ generally correspond to the second term of the Taylor expansion in power of α

$$f_i^\mu(x) = \frac{\partial f^\mu}{\partial \alpha_i}|_{\alpha_i=0} \tag{2.17}$$

$$F_i(x) = \frac{\partial F}{\partial \alpha_i}|_{\alpha_i=0} \ . \tag{2.18}$$

Examples : Parameter-dependent transformations

a) Translation in space-time :

$$x'^\mu = x^\mu + \epsilon^\mu \tag{2.19}$$

$$\phi'(x') = \phi(x) \ . \tag{2.20}$$

The coordinates are transformed, but not the field.

b) Phase transformation :

$$x'^\mu = x^\mu \tag{2.21}$$

$$\phi'(x') = e^{i\alpha}\phi(x) = (1+i\alpha)\phi(x) + \mathcal{O}(\alpha^2) \ . \tag{2.22}$$

The complex field is transformed, but not the coordinates.

The action

$$S = \int_\Omega d^4x \ \mathcal{L}(\phi, \partial_\mu\phi) \tag{2.23}$$

is **invariant** under the transformations (2.15) and (2.16) if $S' = S$ or $\delta S = 0$. Then we say that the system possesses the corresponding **symmetry**. In order to draw the consequences of a symmetry, let us calculate the variation

$$\delta S = \int_{\Omega'} d^4x' \ \mathcal{L}'(\phi', \partial'_\mu\phi') - \int_\Omega d^4x \ \mathcal{L}(\phi, \partial_\mu\phi) \tag{2.24}$$

where the coordinate transformation gives

$$d^4x' = \det\left[\frac{\partial x'^\mu}{\partial x^\nu}\right] d^4x \ . \tag{2.25}$$

At the order $\mathcal{O}(\alpha^2)$, the Jacobian of the transformation becomes

$$\det\left[\frac{\partial x'^\mu}{\partial x^\nu}\right] = 1 + \partial_\rho f_i^\rho \alpha_i + \mathcal{O}(\alpha^2) \ . \tag{2.26}$$

Moreover, still at the order $\mathcal{O}(\alpha^2)$, by use of the chain rule, the derivative transforms as

$$\begin{aligned} \partial'_\mu\phi'(x') &= \partial_\nu\phi'(x')\partial'_\mu x^\nu \\ &= (\partial_\nu\phi + \partial_\nu F_i\alpha_i)(\delta^\nu_\mu - \partial_\mu f_i^\nu\alpha_i) + \mathcal{O}(\alpha^2) \\ &= \partial_\mu\phi + \partial_\mu F_i\alpha_i - \partial_\nu\phi\partial_\mu f_i^\nu\alpha_i + \mathcal{O}(\alpha^2) \end{aligned} \tag{2.27}$$

and its variation becomes

$$\begin{aligned}\delta(\partial_\mu\phi) &= \partial'_\mu\phi'(x') - \partial_\mu\phi(x) \\ &= (\partial_\mu F_i - \partial_\nu\phi\partial_\mu f_i^\nu)\alpha_i + \mathcal{O}(\alpha^2). \end{aligned} \tag{2.28}$$

Thus, the variation of the action gives

$$\begin{aligned}\delta S &= \int_{\Omega'} d^4x'\mathcal{L}' - \int_\Omega d^4x\mathcal{L} \\ &= \int_\Omega d^4x(1+\partial_\mu f_i^\mu\alpha_i)\left[\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \cdots\right] - \int_\Omega d^4x\mathcal{L} \\ &= \int_\Omega d^4x(1+\partial_\mu f_i^\mu\alpha_i)\left[\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}F_i\alpha_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left(\partial_\mu F_i - \partial_\nu\phi\partial_\mu f_i^\nu\right)\alpha_i\right] - \int_\Omega d^4x\mathcal{L} \\ &= \int_\Omega d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}F_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu F_i - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\partial_\mu f_i^\nu + \mathcal{L}\partial_\mu f_i^\mu\right]\alpha_i + \mathcal{O}(\alpha^2). \end{aligned} \tag{2.29}$$

It is possible to collect all the divergences ∂_μ and to show (homework) that the remaining terms cancel each other or are equal to zero by application of the Euler-Lagrange equation. We finally obtain the variation

$$\delta S = \int_\Omega d^4x\ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}F_i - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi f_i^\nu + \mathcal{L}f_i^\mu\right]\alpha_i\ . \tag{2.30}$$

Owing to the invariance $\delta S = 0$, this equation provides a conserved **Noether current**

$$\Theta_i^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}F_i - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi f_i^\nu - \mathcal{L}f_i^\mu\right) \tag{2.31}$$

satisfying the divergence condition or continuity equation

$$\partial_\mu\Theta_i^\mu = 0\ . \tag{2.32}$$

A conserved current implies a conserved charge as it can be seen by integrating the expression (2.32) on a volume V and by applying the divergence theorem

$$\begin{aligned}\int_V d^3r\partial_\mu\Theta_i^\mu &= \int_V d^3r\left[\partial_0\Theta_i^0 + \partial_j\Theta_i^j\right] \\ &= \frac{d}{dt}\int_V d^3r\Theta_i^0 + \int_{\partial V}\boldsymbol{\Theta}_i\cdot d\boldsymbol{\sigma} = 0. \end{aligned} \tag{2.33}$$

If we assume the usual boundary condition

$$\boldsymbol{\Theta}_i|_{\partial V} = 0\ ,$$

we get the **conserved charge**

$$Q_i = \int_V d^3r\ \Theta_i^0 \tag{2.34}$$

satisfying the property

$$\frac{d}{dt}Q_i = 0\ . \tag{2.35}$$

The index i corresponds to the number of parameters of the transformation.

Examples : Application of the Noether's theorem

a) Space-time translation invariance

We consider the transformation given by a constant space-time translation ϵ^μ and by a field ϕ unchanged

$$x'^\mu = x^\mu + \epsilon^\mu \tag{2.36}$$
$$\phi'(x') = \phi(x). \tag{2.37}$$

Then, with $\delta\phi = 0$ ($F_i = 0$) and $x'^\mu = x^\mu + g^{\mu\nu}\epsilon_\nu$ which means $f_i^\nu \to g^{\mu\nu}$, the Noether current (2.31) provides the **energy-momentum**[2] tensor

$$\Theta^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{2.38}$$

which, for a translational invariant system, satisfies the continuity equation

$$\partial_\mu\Theta^{\mu\nu} = 0\ . \tag{2.39}$$

The conserved charge related to this equation is the quantity

$$\begin{aligned} P^\nu &= \int d^3r\Theta^{0\nu} \\ &= \int d^3r\ [\pi\partial^\nu\phi - g^{0\nu}\mathcal{L}] \end{aligned} \tag{2.40}$$

which satisfies the condition

$$\frac{d}{dt}P^\nu = 0\ . \tag{2.41}$$

The first component of P^ν corresponds to the Hamiltonian

$$\begin{aligned} P^0 &= \int d^3r\ [\pi\dot{\phi} - \mathcal{L}] \\ &= H \end{aligned} \tag{2.42}$$

and the three last components to the momentum

$$\mathbf{P} = -\int d^3r\ \pi\nabla\phi \tag{2.43}$$

as we will see later on by using Fourier expansions of the fields $\pi(x)$ and $\phi(x)$.

[2]The energy-momentum tensor $\Theta^{\mu\nu}$ provides, for instance, the coupling of curvature with matter in the Einstein's equations of general relativity

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G\Theta^{\mu\nu}.$$

b) Lorentz invariance

Another well-known example of symmetry follows from the Lorentz invariance of a scalar field. The corresponding infinitesimal transformations read

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}{}_{\nu} x^{\nu} \\ &= \left[\delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu} + \mathcal{O}(\epsilon^2)\right] x^{\nu} \qquad (2.44) \\ \phi'(x') &= \phi(x) \qquad (2.45) \end{aligned}$$

where the matrix $\Lambda = (\Lambda^{\mu}{}_{\nu}) \in \{\Lambda / \Lambda^T g \Lambda = g\}$ is the Lorentz matrix. The conserved charge corresponds (homework) to the angular momentum

$$Q^{jk} = \int d^3r \, [x^j T^{0j} - x^k T^{0j}] \qquad j,k = 1,2,3 \; . \qquad (2.46)$$

where as in (2.43)

$$T^{0j} = -\pi \partial_j \phi \; . \qquad (2.47)$$

For the spinor field $\psi(x)$, we will encounter later on, we must in addition consider the infinitesimal transformation resulting from the spinor field transformation

$$\psi'(x') = S(\Lambda)\psi(x) \; . \qquad (2.48)$$

The calculations (homework) show that the Noether current contains the spin as intrinsic angular momentum. Classical field theory together with group theory could be taught during one semester. For our purposes, the above developments are sufficient.

Chapter 3

The Klein-Gordon Field

3.1 Klein-Gordon Equation

How to describe a physical system represented by a real[1] massive free scalar field $\phi(x)$? Instead of starting with a Lagrangian density that could be derived from general principles of covariance and simplicity, we rather consider an equation similar to the one given by (1.17), the **Klein-Gordon equation**

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = 0 \ . \tag{3.1}$$

This equation written in natural units $\hbar = 1 = c$ is the simplest covariant one, containing a mass m and second derivatives. It can also be derived[2] by interpreting the energy-momentum relation $E^2 = \mathbf{p}^2 + m^2$ as a quantum operator. In another way, by application of the Euler-Lagrange equation (2.5), it is easy to show (do it !) that the Klein-Gordon equation follows from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{3.2}$$

The conjugate field is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{3.3}$$

and allows to write the Hamiltonian

$$\begin{aligned} H &= \int d^3r \left[\pi\dot{\phi} - \mathcal{L}\right] \\ &= \int d^3r \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \ . \end{aligned} \tag{3.4}$$

[1] The choice of a real scalar field is done for reasons of simplicity.

[2] The standard derivation of the Klein-Gordon equation starts from the relativistic energy-momentum relation $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$ and uses the operator correspondences

$$E \to i\hbar \ \frac{\partial}{\partial t} \qquad\qquad \mathbf{p} \to \frac{\hbar}{i}\nabla \ .$$

Then, the resulting operators applied on the complex field $\phi(\mathbf{r}, t)$ provide the equation

$$\left[\Box - (\frac{mc}{\hbar})^2\right]\phi(\mathbf{r}, t) = 0$$

where the notation $\Box \equiv -\partial^\mu\partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ has been introduced.

The Klein-Gordon equation (3.1) is an homogeneous linear differential equation. We seek covariant solutions by considering the four-dimensional Fourier transform[3]

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{-ipx} \tag{3.5}$$

which changes Eq. (3.1) into the eigenvalue equation

$$(p^2 - m^2)\tilde{\phi}(p) = 0. \tag{3.6}$$

We recall the notation $px = p_0 t - \mathbf{p} \cdot \mathbf{r}$. The function $\tilde{\phi}(p)$ is everywhere zero except on the mass shell where $p^2 = m^2$ and must therefore be proportional to a Dirac function[4]

$$\tilde{\phi}(p) = C(p)\ \delta(p^2 - m^2)\ . \tag{3.7}$$

The proportionality factor $C(p)$ can be specified by considering the identity

$$p^2 - m^2 = p_0^2 - \epsilon_{\mathbf{p}}^2 \tag{3.8}$$

where $\epsilon_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$. Then, the well-known relation[5]

$$\delta(p^2 - m^2) = \frac{1}{2\epsilon_{\mathbf{p}}} \left[\delta(p_0 - \epsilon_{\mathbf{p}}) + \delta(p_0 + \epsilon_{\mathbf{p}})\right] \tag{3.9}$$

shows that p_0 only takes the two values $p_0 = \pm\epsilon_{\mathbf{p}}$ and allows for a separation of $C(p)$ into two parts, one with respect to $p_0 > 0$ and the other with respect to $p_0 < 0$, each one depending on the vector $\mathbf{p}$ only

$$C(p) = 2\pi \left\{ \begin{array}{lll} a(\mathbf{p}) & \text{for} & p_0 > 0 \\ b(\mathbf{p}) & \text{for} & p_0 < 0 \end{array} \right. = 2\pi \Big[\theta(+p_0) a(\mathbf{p}) + \theta(-p_0) b(\mathbf{p})\Big]\ . \tag{3.10}$$

The step function $\theta(p_0)$ and a conventional 2π factor have been introduced. Moreover, the real scalar field condition $\phi(x) = \phi^*(x)$ yields the relation

$$\tilde{\phi}(p) = \tilde{\phi}^*(-p)\ , \tag{3.11}$$

which contains the function

$$\tilde{\phi}^*(-p) = 2\pi \Big[\theta(-p_0) a^*(-\mathbf{p}) + \theta(p_0) b^*(-\mathbf{p})\Big] \delta(p^2 - m^2)$$

[3]The physical system we are considering here is infinite and therefore requires Fourier transform. It can be compared to the finite periodic system of the vibrating string of size ℓ given in chapter 1.2, where we have used Fourier series. But, we know that in the limit $\ell \to \infty$ the Fourier series becomes a Fourier transform through the replacement $\sum_{k=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} \ell dp$ where we have set $p = k/\ell$ for ℓ large.

[4]To be precise, one should use the concept of Dirac distribution or linear functional on the Schwartz space, $D_{x_0} : \{f\} \to \mathbb{R}$ defined as $D_{x_0}[f] = f(x_0)$.
Then the development $(xD_{x_0})[f] = D_{x_0}[xf] = x_0 f(x_0) = x_0 D_{x_0}[f]$, leads to the eigenvalue equation $(x - x_0)D_{x_0}[f] = 0$. The Dirac functional $D_{x_0}[f] = f(x_0)$ is the correct mathematical expression of the Dirac "function" symbolically written $\delta(x - x_0)$ or $\int \delta(x - x_0) f(x)\ dx = f(x_0)$.

[5]For a proof of this relation, we consider a function $u(x)$ with zeros at the points $x_i \in \mathbb{R}$ and take tiny intervals $[-\epsilon_i, \epsilon_i]$ around these zeros. Then, in each of these interval, we make a change of variable $y = u(x)$, $dy = u'(x)dx$ and obtain $\int_{-\infty}^{+\infty} \delta(u(x)) f(x)\ dx = \sum_i \int_{-\epsilon_i}^{+\epsilon_i} \delta(y)\ f(x(y))\ \frac{dy}{|u'(x(y))|} = \sum_i \frac{1}{|u'(x_i)|} f(x_i) = \int_{-\infty}^{+\infty} \sum_i \frac{1}{|u'(x_i)|} \delta(x - x_i)\ f(x)\ dx$.

and allows to deduce that $b^*(-\mathbf{p}) = a(\mathbf{p})$. Then, the field becomes

$$\tilde{\phi}(p) = 2\pi\Big[\theta(p_0)a(\mathbf{p}) + \theta(-p_0)a^*(-\mathbf{p})\Big]\delta(p^2 - m^2) \ . \tag{3.12}$$

The integration of (3.5) over dp_0 can be easily performed and provides the expression

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\int_{-\infty}^{+\infty} dp_0\Big[\theta(p_0)a(\mathbf{p}) + \theta(-p_0)a^*(-\mathbf{p})\Big]\delta(p_0^2 - \epsilon_{\mathbf{p}}^2)e^{-ipx} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{p}}}\left[a(\mathbf{p})\int_0^{+\infty}\delta(p_0 - \epsilon_{\mathbf{p}})e^{-ipx}dp_0 + a^*(-\mathbf{p})\int_{-\infty}^{0}\delta(p_0 + \epsilon_{\mathbf{p}})e^{-ipx}dp_0\right] \\
&= \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\left[a(\mathbf{p})\ e^{-i(\epsilon_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{r})} + a^*(\mathbf{p})\ e^{i(\epsilon_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{r})}\right]
\end{aligned}$$

that leads, with the obvious notation $px = \epsilon_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{r}$, to the formula for a free classical Klein-Gordon field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\left[a(\mathbf{p})\ e^{-ipx} + a^*(\mathbf{p})\ e^{ipx}\right] \ . \tag{3.13}$$

The word free means that the field is not subject to external forces and is non-interacting with other fields. Moreover, the above calculations show that, despite a three-dimensional integration, the expression (3.13) is still Lorentz-invariant since it is identical to the Fourier transform (3.5). We therefore conclude that the quantity

$$\frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \tag{3.14}$$

is a **Lorentz-invariant measure**. Actually, this procedure of covariant Fourier expansion will be implicitly applied to the other fields we will encounter later on. A question to ask oneself : what is the expansion form for a complex scalar field ? The answer can be found by looking at formula (3.11) and (3.13).

If we want to extend the study of the relativistic scalar field to non-fre field, we can add a source density $\rho(x)$ to the Klein-Gordon equation

$$(\Box - m^2)\phi(x) = \rho(x) \ . \tag{3.15}$$

With the **Feynman propagator** $G_F(x-x')$ or **Green's function** defined by the equation

$$(\Box - m^2)G_F(x - x') = \delta^{(4)}(x - x') \ , \tag{3.16}$$

the solution of Eq. (3.15) can be written

$$\phi(x) = \int d^4x' G_F(x - x')\ \rho(x') \ . \tag{3.17}$$

Then, the four-dimensional Fourier transform

$$G_F(x - x') = \int \frac{d^4p}{(2\pi)^4}\ \tilde{G}(p)\ e^{-ip(x-x')} \tag{3.18}$$

allows to write Eq. (3.16) in the algebraic form $(p^2 - m^2)\tilde{G}(p) = 1$ which provides the Feynman propagator in p-space

$$\tilde{G}(p) = \frac{1}{p^2 - m^2} \ . \tag{3.19}$$

By inserting $\tilde{G}(p)$ into (3.18), we arrive at the four-dimensional integral

$$G_F(x - x') = \int \frac{d^4p}{(2\pi)^4} \ \frac{e^{-ip(x-x')}}{(p^0)^2 - (\mathbf{p}^2 + m^2)} \tag{3.20}$$

whose solution is not unique. The integrand has two poles at $p^0 = \pm\epsilon_{\mathbf{p}}$.

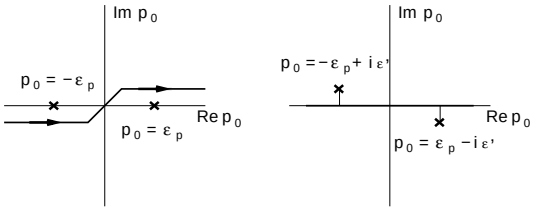

Figure 3.1: Integration path

These poles can be avoided by a convenient choice of integration paths in the complex p^0-plane. On the path depicted in Figure 3.1, the integral (3.20) takes the form

$$G_F(x - x') = \int \frac{d^4p}{(2\pi)^4} \ \frac{e^{-ip(x-x')}}{p^2 - m^2 + i\epsilon} \tag{3.21}$$

where the parameter ϵ displays the infinitesimal displacements[6] of the poles in the upper and lower half-plane respectively, and thus indicates the choice of the path. As asserted by the Jordan's lemma[7], the integration over dp^0 can be closed by large semi-circles of zero contribution, in the upper half-plane if $t - t' < 0$ and in the lower half-plane if $t - t' > 0$. This lemma thus shows that the integrals on the closed paths are equal to those along the real axis. Finally, the application of the residue theorem[8] gives the Feynman propagator in x-configuration

$$G_F(x - x') = -i \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \left[\theta(t - t') \ e^{-ip(x-x')} + \theta(t' - t) \ e^{ip(x-x')}\right] \tag{3.22}$$

[6]The replacement $(\mathbf{p}^2 + m^2) \to (\mathbf{p}^2 + m^2 - i\epsilon)$, $\epsilon \to 0^+$ gives the poles $p^0 = \pm(\epsilon_{\mathbf{p}} - i\epsilon')$.

[7]Jordan's lemma : If $|f(z)| \longrightarrow 0$ for $|z| \to \infty$, then

$$\int_{\cap} f(z)e^{i\lambda z} \longrightarrow 0 \ \text{ for } \ \Omega \to \infty, \lambda > 0 \qquad\qquad \int_{\cup} f(z)e^{i\lambda z} \longrightarrow 0 \ \text{ for } \ \Omega \to \infty, \lambda < 0$$

where Ω is the radius of the semi-circles $\cap$ and $\cup$ drawn in the upper and lower half-plane respectively.

[8]Residue theorem : The closed integral of a meromorphic function f gives

$$\oint f(z)dz = 2\pi i \sum Res\, f$$

for a counter-clockwise integration. The quantity $Res\, f$ is the residue of the function f.

where $px = \epsilon_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{r}$ and where $\theta(t)$ is the step function. The calculation of the residues[9] of the single poles $p^0 = \pm\epsilon_{\mathbf{p}}$ has been done in the usual way. Other possible choices of paths can be accepted or rejected in accordance with appropriate physical conditions[10]. As a particular example, we come back to a three-dimensional system and consider a local source of the form

$$\rho(\mathbf{r}) = -g\delta(\mathbf{r}-\mathbf{r}') \tag{3.23}$$

where g is a given coupling. Then, Eq. (3.15) is replaced by the stationary equation

$$(\nabla^2 - m^2)\phi(\mathbf{r}-\mathbf{r}') = -g\delta(\mathbf{r}-\mathbf{r}') \tag{3.24}$$

whose solution can be calculated (homework) by means of a three-dimensional Fourier transform which yields the well-known **Yukawa potential**

$$\phi(|\mathbf{r}-\mathbf{r}'|) = \frac{g}{4\pi}\,\frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{3.25}$$

This short-range potentiel should be compared to the the Coulomb potential $\Phi(|\mathbf{r}-\mathbf{r}'|) = q/(4\pi|\mathbf{r}-\mathbf{r}'|)$ which is the solution of the equation $\nabla^2\Phi(\mathbf{r}-\mathbf{r}') = -q\delta(\mathbf{r}-\mathbf{r}')$.

Up to now in this chapter no reference to quantum mechanics[11] has been done. The scalar field $\phi(x)$ has been treated as a pure classical field. The next step toward relativistic quantum mechanics could attempt to transform Eq. (3.15) into a system of two first-order in time differential equations (why ?) and to calculate, for instance, probability amplitudes. An explicit calculation of the scattering of a relativistic electron by the Coulomb potential will be done in section 5.2. For the moment, we pursue one of the main goals of this course, the field quantization.

3.2 Quantization of the Klein-Gordon Field

The scalar field quantization uses the considerations of section 1.3. The Fourier coefficients $a(\mathbf{p})$ are replaced by operators acting on the Fock space. In particular, the complex conjugate is replaced by the adjoint operator

$$a^*(\mathbf{p}) \quad \rightarrow \quad a^\dagger(\mathbf{p})\ . \tag{3.26}$$

Then, the Fourier expansion (3.13) becomes the **Klein-Gordon field operator**

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\left[a(\mathbf{p})e^{-ipx} + a^\dagger(\mathbf{p})e^{ipx}\right] \tag{3.27}$$

[9] Residue of a single pole :

$$\begin{aligned} Res_\pm\left[\frac{e^{-p(x-x')}}{(p^0-\epsilon_{\mathbf{p}})(p^0+\epsilon_{\mathbf{p}})}\right] &= \lim_{p^0\to\pm\epsilon_{\mathbf{p}}}(p^0\mp\epsilon_{\mathbf{p}})\frac{e^{-p^0(t-t')+\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{(p^0-\epsilon_{\mathbf{p}})(p^0+\epsilon_{\mathbf{p}})} \\ &= \pm\frac{e^{\mp\epsilon_{\mathbf{p}}(t-t')+\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{2\epsilon_{\mathbf{p}}}\ . \end{aligned}$$

[10] This choice of path corresponds to the time-ordered product we will introduce in section 6.4.

[11] The quantum correspondence relations between energy and time derivative, between momentum and spatial derivatives can be ignored since the Klein-Gordon equation can be directly derived from the classical Lagrangian density (3.2).

and yields also the conjugate field operator

$$\pi(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \left[a(\mathbf{p})e^{-ipx} - a^\dagger(\mathbf{p})e^{ipx}\right](-i\epsilon_{\mathbf{p}}) \ . \tag{3.28}$$

These two scalar field operators are the basic quantities of QFT. Their SI units can be recovered by a dimensional analysis[12]. Field quantization requires, as in (1.40) and (1.39), the equal-time canonical commutation relations

$$[\phi(\mathbf{r},t), \pi(\mathbf{r}',t)] = i\delta(\mathbf{r}-\mathbf{r}') \tag{3.29}$$

$$[\phi(\mathbf{r},t), \phi(\mathbf{r}',t)] = 0 = [\pi(\mathbf{r},t), \pi(\mathbf{r}',t)] \tag{3.30}$$

from where we deduce (homework)

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{p}')\right] = (2\pi)^3 2\epsilon_{\mathbf{p}}\ \delta(\mathbf{p}-\mathbf{p}') \tag{3.31}$$

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 = \left[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')\right] \ . \tag{3.32}$$

The operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ are interpreted respectively as **creation** and **annihilation** operators of a particle of momentum $\mathbf{p}$, called **boson**. Applied on the vacuum state $|0\rangle$, they give

$$a^\dagger(\mathbf{p})|0\rangle = |\mathbf{p}\rangle \qquad a(\mathbf{p})|0\rangle = 0 \ . \tag{3.33}$$

Starting from the vacuum normalization $\langle 0|0\rangle = 1$ and using the commutations relations (3.31), we deduce the one-particle state normalization

$$\begin{aligned} \langle \mathbf{p}|\mathbf{p}'\rangle &= \langle 0|a(\mathbf{p})a^\dagger(\mathbf{p}')|0\rangle \\ &= \langle 0|a^\dagger(\mathbf{p}')a(\mathbf{p})|0\rangle + \langle 0|(2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')|0\rangle \\ &= (2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}'). \end{aligned} \tag{3.34}$$

With the tensor product of two single states, we can define a two-particle state

$$|\mathbf{p}_1\rangle \otimes |\mathbf{p}_2\rangle = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle \otimes |0\rangle \ . \tag{3.35}$$

Because the operators $a^\dagger(\mathbf{p})$ commute among themselves, the state is **symmetric** under exchange of any two particles. A n-particle state[13] of momenta $\mathbf{p}_i$ is represented by the tensor product of individual kets and is written

$$|\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\rangle = |\mathbf{p}_1\rangle \otimes |\mathbf{p}_2\rangle \otimes \cdots \otimes |\mathbf{p}_n\rangle \ . \tag{3.36}$$

[12]The conversion of the scalar field (3.27) into SI units is done by a dimensional analysis as explained at the beginning of this course (7). From the Lagrangian density (3.2), we deduce the SI units of the field $[\phi] = (ML^2T^{-2})^{1/2}L^{-1/2}$. Then, the four-dimensional Fourier tranform gives $[\tilde{\phi}] = (ML^2T^{-2})^{1/2}L^{7/2}$ and formula (3.7), containing a δ function depending on the square energy, shows that $[a] = (ML^2T^{-2})^{5/2}L^{7/2}$. With $\epsilon_{\mathbf{p}} = \sqrt{(mc^2)^2 + (pc)^2}$ and a dimensionless argument $px/\hbar$, the field expression (3.27) leads to the dimensional relation $1 = (MLT^{-1})^3(ML^2T^{-2})L^4(ML^2T^{-1})^\alpha(LT^{-1})^\beta$ from where we deduce $\alpha = -4$ and $\beta = -1$. Finally, in SI units, formula (3.27) takes the form

$$\phi(x) = \frac{1}{\hbar c}\int \frac{d^3p}{(2\pi\hbar)^3 2\epsilon_{\mathbf{p}}} \left[a(\mathbf{p})e^{-ipx/\hbar} + a^\dagger(\mathbf{p})e^{ipx/\hbar}\right] \ .$$

[13]For reasons of simplicity, we drop the occupation number representation $|n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, \cdots\rangle$ of section 1.3.

The action of $a^\dagger(\mathbf{p})$ on this state creates a new particle of momentum $\mathbf{p}$

$$a^\dagger(\mathbf{p})|\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle = |\mathbf{p},\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle \tag{3.37}$$

and the action of $a(\mathbf{p})$ follows from the definition of the adjoint

$$\langle \mathbf{p}'_2,\cdots,\mathbf{p}'_n|a(\mathbf{p})|\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle = \langle \mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n|a^\dagger(\mathbf{p})|\mathbf{p}'_2,\cdots,\mathbf{p}'_n\rangle^* \ . \tag{3.38}$$

Actually, it is possible to establish (homework) the bracket relation

$$\begin{aligned}&\langle \mathbf{p}'_1,\mathbf{p}'_2,\cdots,\mathbf{p}'_n|\mathbf{p},\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle \\ &\qquad = (2\pi)^3\sum_{i=1}^n 2\epsilon_{\mathbf{p}'_i}\delta(\mathbf{p}-\mathbf{p}'_i)\langle \mathbf{p}'_1,\cdots,\hat{\mathbf{p}}'_i,\cdots,\mathbf{p}'_n|\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle,\end{aligned} \tag{3.39}$$

where the symbol $\hat{}$ means that the momenta $\hat{\mathbf{p}}'_i$ must be ignored. Then, from this expression and from the definition (3.38) of the adjoint, we deduce the action of the annihilation operator

$$a(\mathbf{p})|\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle = (2\pi)^3\sum_{i=1}^n 2\epsilon_{\mathbf{p}_i}\delta(\mathbf{p}-\mathbf{p}_i)|\mathbf{p}_1,\cdots,\hat{\mathbf{p}}_i,\cdots,\mathbf{p}_n\rangle. \tag{3.40}$$

The **number operator** N is defined as

$$N = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} a^\dagger(\mathbf{p})a(\mathbf{p}) \tag{3.41}$$

and its action on some general state gives the eigenvalue equation (check it !)

$$N|\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle = n|\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n\rangle \tag{3.42}$$

where n is the number of particles. The Klein-Gordon Hamiltonian expressed in terms of the operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ takes the form (homework)

$$\begin{aligned}H &= \frac{1}{2}\int d^3r\Big(\pi^2+(\nabla\phi)^2+m^2\phi^2\Big) \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\ \epsilon_{\mathbf{p}}\Big[a(\mathbf{p})^\dagger a(\mathbf{p})+a(\mathbf{p})a(\mathbf{p})^\dagger\Big]\ .\end{aligned} \tag{3.43}$$

Then, the commutation relations (3.31) yields the Hamiltonian operator

$$H = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\ \epsilon_{\mathbf{p}}\Big[a(\mathbf{p})^\dagger a(\mathbf{p})+\epsilon_{\mathbf{p}}(2\pi)^3\delta(0)\Big] \tag{3.44}$$

which gives an infinite[14] energy on the vacuum state $|0\rangle$. The energy normalization is given as usual by the normal product (1.62) which allows to replace the Hamiltonian (3.43) by

[14]In order to see that H is infinite on the vacuum state $|0\rangle$, we use the following formal identity

$$(2\pi)^3\delta(0) = \lim_{\mathbf{k}\to\mathbf{k}'}\int_V d^3r\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = V$$

where the limit of infinite volume should be taken at the end of the calculation. For a cut off $p=\Lambda$, integral (3.44) applied on the vacuum gives a quartically diverging energy

$$H|0\rangle = V\ \frac{1}{4(2\pi)^2}\Lambda^4\left[1+o(\frac{m^2}{\Lambda^2})\right]|0\rangle \qquad \Lambda\to\infty\ .$$

the finite Hamiltonian

$$:H: = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \, \epsilon_{\mathbf{p}} \, a(\mathbf{p})^\dagger a(\mathbf{p}) \; . \tag{3.45}$$

At this point, we come back to the classical Klein-Gordon field whose translational invariance provides the conserved charge

$$\mathbf{P} = -\int d^3r \, \pi \nabla \phi \tag{3.46}$$

defined in (2.43). With the field operators (3.27) and (3.28), it can be written in term of creation and annihilation operators and leads (homework) to the momentum operator

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \, \mathbf{p} \; a^\dagger(\mathbf{p}) a(\mathbf{p}) \tag{3.47}$$

and the corresponding eigenvalue equation

$$\mathbf{P}|\mathbf{p}'\rangle = \mathbf{p}'|\mathbf{p}'\rangle \; . \tag{3.48}$$

To conclude this section concerning the quantization of the Klein-Gordon field, it is important to emphasize that in quantum field theory the quantity $\phi(\mathbf{r}, t)$ is an operator whereas $\mathbf{r}$ is a parameter or an index. This has to be compared with the single-particle quantum theory, where $\phi(\mathbf{r}, t)$ is the wave function specifying the state of the system, but the position coordinate $\mathbf{r}$ is an operator. In the next chapters, we will adopt the same procedure for defining the electromagnetic and the fermionic quantum field. But, still as non-interacting quantum fields.

Chapter 4

The Electromagnetic Field

4.1 Maxwell's Equations and Gauge Field

The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are governed by the Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{B}}{\partial t} = \mathbf{j} \tag{4.1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} = 0 \tag{4.2}$$

and are connected to the gauge field $A^\mu = (\Phi, \mathbf{A})$ by the relations

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{4.3}$$

that follow from the Poincaré's lemma. The electromagnetic **field tensor** defined as a function of the four-vector field $A^\mu(x)$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.4}$$

is invariant with respect to the gauge transformation

$$A'^\mu = A^\mu + \partial^\mu \chi \ . \tag{4.5}$$

For a free field ($j^\mu = 0$), the elecromagnetic-field Lagrangian density corresponding to the simplest invariant contraction of the field tensor can be written[1] as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \ . \end{aligned} \tag{4.6}$$

Then, the field equations

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} \right) - \frac{\partial \mathcal{L}}{\partial A_\beta} = 0 \ ,$$

yield (homework) the source-free Maxwell's equations

$$\partial^\alpha F_{\alpha\beta} = 0 \ . \tag{4.7}$$

[1] In standard units we have $\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 - \frac{1}{\mu_0}\mathbf{B}^2)$ with $c^2 = \frac{1}{\epsilon_0 \mu_0}$.

The conjugate momenta are given by

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \tag{4.8}$$

$$\pi^j = \frac{\partial \mathcal{L}}{\partial \dot{A}_j} = -F^{0j} = E^j \qquad j = 1,2,3 \tag{4.9}$$

and the Hamiltonian takes the form

$$\begin{aligned} H &= \int d^3r \left[\pi^\mu \dot{A}_\mu - \mathcal{L}\right] \\ &= \frac{1}{2} \int d^3r (\mathbf{E}^2 + \mathbf{B}^2) \end{aligned} \tag{4.10}$$

where we made use of the equation for a free electric field $\nabla \cdot \mathbf{E} = 0$. The Maxwell's equations (4.7) are explicitly written

$$\partial^\mu \partial_\mu A_\nu - \partial_\nu (\partial^\mu A_\mu) = 0 \tag{4.11}$$

and, with the Lorentz gauge condition $\partial^\mu A_\mu = 0$, yield the wave equation

$$\Box A^\mu = 0 \ . \tag{4.12}$$

The resolution of this vector equation can be done by using the same arguments (do it!) as for the scalar field. The four-dimensional Fourier transform of $A^\mu(x)$ gives the eigenvalue equation $k^2 \tilde{A}^\mu(k) = 0$ whose solutions $\tilde{A}^\mu(k) = C^\mu(k)\, \delta(k^2)$ leads to a field expression similar to (3.13). However, the Lorentz gauge condition applied to plane-wave solutions of the form

$$A^\mu(x) \sim a^\mu(k)\ e^{-ikx} \tag{4.13}$$

implies the orthogonality relation

$$k_\mu a^\mu = 0 \ . \tag{4.14}$$

It follows that three of the four components of the four-vector $[a^\mu(k)]$ are independant. We are then left with one more degree of freedom than the number of polarizations[2]. It is possible to avoid this additional degree of freedom by using the **Coulomb gauge** conditions[3]

$$\nabla \cdot \mathbf{A} = 0 \qquad \Phi = 0. \tag{4.15}$$

These two conditions limit the number of field degrees of freedom to 2 and allow for a simple physical interpretation. But, they have the drawback to hide the explicit relativistic covariance. For each value of the wave vector $\mathbf{k}$, we describe the degrees of freedom with two unit vectors $\mathbf{e}_\alpha(\mathbf{k})$, $\alpha = 1,2$ called **polarization vectors** and generally written

$$[e^\mu_\alpha(\mathbf{k})] = \begin{bmatrix} 0 \\ \mathbf{e}_\alpha(\mathbf{k}) \end{bmatrix} \qquad \alpha = 1,2 \ . \tag{4.16}$$

[2]This situation, which displays one unphysical degree of freedom, does not appear in the basic course of electrodynamics where the calculations are performed with the $\mathbf{E}$ and $\mathbf{B}$ fields (without gauge field).

[3]The choice of this Coulomb gauge is possible in free space (*i.e.* for $j^\mu = 0$). Explicitly, it must be shown (homework) that for a given A_μ, there exists a $A'_\mu = A_\mu + \partial_\mu \chi$ such that $\nabla \cdot \mathbf{A}' = 0$ and $A'_0 = 0$.

They have the properties

$$\mathbf{e}_\alpha(\mathbf{k}) \cdot \mathbf{e}_{\alpha'}(\mathbf{k}) = \delta_{\alpha\alpha'} \qquad \mathbf{k} \cdot \mathbf{e}_\alpha(\mathbf{k}) = 0 \qquad \mathbf{e}_\alpha(-\mathbf{k}) = (-1)^\alpha \mathbf{e}_\alpha(\mathbf{k}) \ . \tag{4.17}$$

The first relation is obvious, the second follows from the gauge condition $\nabla \cdot \mathbf{A} = 0$ and the last can be checked by a look at Figure 4.1. We finally arrive at the free electromagnetic gauge field[4]

$$\mathbf{A}(x) = \int \frac{d^3k}{(2\pi)^3 2\epsilon_\mathbf{k}} \sum_{\alpha=1}^{2} \mathbf{e}_\alpha(\mathbf{k}) \left[a_\alpha(\mathbf{k}) e^{-ikx} + a^*_\alpha(\mathbf{k}) e^{ikx} \right] \tag{4.18}$$

where $a_\alpha(\mathbf{k})$, $a^*_\alpha(\mathbf{k})$ $\alpha = 1, 2$ are the Fourier coefficients. The argument of the exponential function is written $kx = \epsilon_\mathbf{k} t - \mathbf{k} \cdot \mathbf{r}$ and the energy $\epsilon_\mathbf{k} = |\mathbf{k}| = \omega_\mathbf{k}$ is fixed by the dispersion relation $k^2 = 0$.

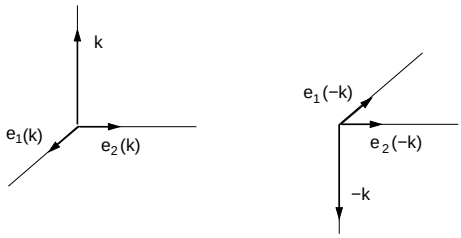

Figure 4.1: Polarization vectors

We can extend the discussion by considering the presence of a current $j^\mu(x) = (\rho, \mathbf{j})$. Then the Maxwell's equations

$$\partial_\mu F^{\mu\nu} = j^\nu$$

subject to the Lorenz gauge condition $\partial^\mu A_\mu = 0$ yield the inhomogeneous equation

$$\Box A^\mu(x) = -j^\mu(x) \ . \tag{4.19}$$

The solution of this equation can take the integral form

$$A^\mu(x) = -\int d^4x' D_F(x - x') \ j^\mu(x') \tag{4.20}$$

where the Green's function $D_F(x - x')$ relative to the operator $\Box$ obeys the equation

$$\Box D_F(x - x') = \delta^{(4)}(x - x') \ . \tag{4.21}$$

By Fourier transforming this equation, we arrive (homework) at the solution

$$D_F(x - x') = \frac{1}{(2\pi)^4} \int d^4k \ e^{-ik(x-x')} (-\frac{1}{k^2}) \ .$$

The result of this integral is put into Eq. (4.20) and provides, by application of the causality principle, the retarded potential of classical electrodynamics

$$A^\mu_{ret}(x) = \frac{1}{4\pi} \int d^3r' \, dt' \, \frac{\delta\left(t' + \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t\right)}{|\mathbf{r} - \mathbf{r}'|} \ j^\mu(x') \ . \tag{4.22}$$

[4]The field $\mathbf{A}(x)$ is real since it is connected to the real $\mathbf{E}$ and $\mathbf{B}$ fields by (4.3).

4.2 Coulomb Gauge Quantization

Electrodynamics is a genuine field theory since its quantization can only be formulated as a quantum field theory[5]. The quantization process for the gauge field $A^\mu(x)$ first considers the classical field equation (4.12), it then identifies the conjugate momenta and finally requires quantum canonical commutation relations for the field operators. Unfortunately, the component A_0 of the gauge field does not have a conjugate momentum since π^0 given by (4.8) vanishes identically. We are thus faced with a new difficulty in performing covariant electromagnetic field quantization. But, we already know how to circumvent this problem. Instead of the Lorentz gauge condition, we apply the Coulomb gauge conditions (4.15) and derive the classical field expression (4.18). Then, the replacement of the Fourier coefficients by operators yields the **electromagnetic field operator**

$$\mathbf{A}(x) = \int \frac{d^3k}{(2\pi)^3 2\epsilon_{\mathbf{k}}} \sum_{\alpha=1}^{2} \mathbf{e}_\alpha(\mathbf{k}) \left[a_\alpha(\mathbf{k}) e^{-ikx} + a_\alpha^\dagger(\mathbf{k}) e^{ikx} \right] \tag{4.23}$$

where $a_\alpha^\dagger(\mathbf{k})$, $a_\alpha(\mathbf{k})$ are the photon creation and annihilation operators of each polarization $\mathbf{e}_\alpha(\mathbf{k})$, $\alpha = 1, 2$. The argument of the exponential function is written $kx = \epsilon_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r}$ and the dispersion relation leads to the photon energy $\epsilon_{\mathbf{k}} = |\mathbf{k}| = \omega_{\mathbf{k}}$. Now, if we consider the three non-zero components of the conjugate field

$$\pi^j = -F^{0j} = -\dot{A}^j = E^j \qquad j = 1, 2, 3 \ , \tag{4.24}$$

the equal-time canonical commutation relation can be written

$$\left[A_{j'}(\mathbf{r}', t), \pi^j(\mathbf{r}, t) \right] = i\delta^j_{j'}\, \delta(\mathbf{r} - \mathbf{r}') \ . \tag{4.25}$$

With the value (4.24) of π^j, the left member takes also the form

$$\left[\pi^j(\mathbf{r}, t), A_{j'}(\mathbf{r}', t) \right] = \left[E^j(\mathbf{r}, t), A_{j'}(\mathbf{r}', t) \right] \ . \tag{4.26}$$

But these two last relations are not consistent with the Maxwell's equation

$$\nabla \cdot \mathbf{E} \equiv \partial_j E^j = 0 \ ,$$

since the application of the divergence on both sides of(4.25) brings out the contradiction

$$\begin{aligned} \partial_j \left[\pi^j(\mathbf{r}, t), A_{j'}(\mathbf{r}', t) \right] &= \partial_j E^j(\mathbf{r}, t) A_{j'}(\mathbf{r}', t) \ - A_{j'}(\mathbf{r}', t) \partial_j E^j(\mathbf{r}, t) = 0 \\ &= -i\ \delta^j_{j'} \partial_j \delta(\mathbf{r} - \mathbf{r}') = -i\ \partial_{j'} \delta(\mathbf{r} - \mathbf{r}') \neq 0 \end{aligned}$$

How to get rid of it ? By considering the derivative of $\delta(\mathbf{r} - \mathbf{r}')$ with respect to x_j

$$\partial_j \delta(\mathbf{r} - \mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\ ip_j \neq 0 \ ,$$

we see that the new defined delta function

$$\delta^{tr}_{jj'}(\mathbf{r} - \mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \left[\delta_{jj'} - \frac{p_j p_{j'}}{\mathbf{p}^2} \right] \tag{4.27}$$

[5] In the basic course of quantum mechanics, photons are used as phenomenological objects for discussing the concept of quantum state, but they do not appear anymore afterwards.

has a zero divergence (sum over j !)

$$\partial^j \delta^{tr}_{jj'}(\mathbf{r}-\mathbf{r}') = i \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} p^j \left[\delta_{jj'} - \frac{p_j p_{j'}}{\mathbf{p}^2}\right] = 0 \ .$$

Thus, in order to stay consistent with the Maxwell's equation $\nabla \cdot \mathbf{E} = 0$, we consider the new quantum commutation relation

$$[A_j(\mathbf{r},t), \pi_{j'}(\mathbf{r}',t)] = -i \ \delta^{tr}_{jj'}(\mathbf{r}-\mathbf{r}') \ . \tag{4.28}$$

From this relation, we can deduce[6] (homework) the commutation relations for the creation and annihilation operators

$$\left[a_\alpha(\mathbf{k}), a^\dagger_{\alpha'}(\mathbf{k}')\right] = (2\pi)^3 2\omega_{\mathbf{k}} \ \delta(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'} \tag{4.29}$$

$$\left[a_\alpha(\mathbf{k}), a_{\alpha'}(\mathbf{k}')\right] = 0 = \left[a^\dagger_\alpha(\mathbf{k}), a^\dagger_{\alpha'}(\mathbf{k}')\right] \qquad \alpha, \alpha' = 1, 2. \tag{4.30}$$

The Fock space is defined as in section 3.2. It contains photon states given by the tensor product $|n_{k_1}, \alpha_1\rangle \otimes \cdots \otimes |n_{k_j}, \alpha_j\rangle \otimes \cdots$ of single states, where the n_{k_j} are the occupation numbers and the $\alpha_j = 1, 2$ represent the two polarizations. The covariant quantization of the electromagnetic field is much more tricky, interested students can consult textbooks.

4.3 Spontaneous Emission

It is well known that an atom in an excited state can spontaneously emit radiation and return to its ground state. The phenomenon is not predicted by a semi-classical quantum theory that only describes the atomic state changes corresponding to absorption or to stimulated emission. These changes are generated by an external classical electromagnetic field. In order to explain spontaneous transitions, the electromagnetic field $\mathbf{A}$ must be interpreted as a quantum field. The electron in an electromagnetic field can be described by the Hamiltonian

$$H = :H_{ph}: + \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 + V(\mathbf{r}) \tag{4.31}$$

where the first term is the normal ordered free photon Hamiltonian

$$:H_{ph}: = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \ \omega_{\mathbf{k}} \sum_{\alpha=1}^{2} a^\dagger_\alpha(\mathbf{k}) a_\alpha(\mathbf{k}) \ , \tag{4.32}$$

the second term represents a single charge $-e$ in an electromagnetic field $\mathbf{A}$ and the third term is the atomic potential. Owing to the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$, the operators $\mathbf{p}$ and $\mathbf{A}$ commute. This property is used for expanding the second term of (4.31). By neglecting[7] the quadratic term $\mathbf{A}^2$, we can separate the Hamiltonian (4.31) into photonic, atomic and interacting parts

$$\begin{aligned} H &= :H_{ph}: + \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) + \frac{e}{m}\mathbf{A}\cdot\mathbf{p} \\ &= :H_{ph}: + H_{at} + H_I \end{aligned} \tag{4.33}$$

[6] In order to do that, we first invert the field (4.23) and its conjugate by means of scalar multiplications $\int d^3r' e^{i\mathbf{p}\cdot\mathbf{r}'} \mathbf{e}_\alpha(\mathbf{p})\cdot\mathbf{A}(\mathbf{r}')$, $\int d^3r' e^{i\mathbf{p}\cdot\mathbf{r}'} \mathbf{e}_\alpha(\mathbf{p})\cdot\boldsymbol{\pi}(\mathbf{r}')$, and then we calculate the commutator between creation and annihilation operators.

[7] This approximation is valid insofar as the amplitude of the electromagnetic field is not to large.

where the atomic and interaction Hamiltonians are defined as

$$H_{at} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) \tag{4.34}$$

$$H_I = \frac{e}{m}\mathbf{A}\cdot\mathbf{p}\ . \tag{4.35}$$

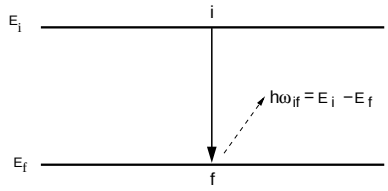

Figure 4.2: Initial and final states of the spontaneous emission

The electromagnetic field $\mathbf{A}$ is quantized, whereas the atomic Hamiltonian represents a one-electron quantum system. As we know from the time-dependant perturbation theory, the first order transition amplitude $c_{fi}^{(1)}$ between an initial state $|i\rangle$ and a final state $|f\rangle$ is given by the expression

$$c_{fi}^{(1)} = \frac{1}{i\hbar}\int_0^T dt\ e^{-i\omega_{if}t}\left(\frac{e}{m}\langle f|\mathbf{A}\cdot\mathbf{p}|i\rangle\right) \tag{4.36}$$

containing the frequency

$$\omega_{if} = \frac{1}{\hbar}(E_i - E_f) \tag{4.37}$$

of energies E_i, E_f illustrated in Figure 4.2. The calculation of the expectation value appearing in (4.36) requires the knowledge of free initial and final states. The commuting operators H_{at} and $:H_{ph}:$ of the total Hamiltonian allow to write these free states as the tensor product of atomic and photonic states

- initial state : $|i\rangle = |E_i\rangle \otimes |0\rangle$
- final state : $|f\rangle = |E_f\rangle \otimes |\mathbf{k},\alpha\rangle \qquad \alpha = 1,2$

where the single states obey the relations

$$H_{at}|E_\lambda\rangle = E_\lambda|E_\lambda\rangle \qquad \lambda = i,f \tag{4.38}$$

$$a_\alpha^\dagger(\mathbf{k})|0\rangle = |\mathbf{k},\alpha\rangle \qquad \alpha = 1,2\ . \tag{4.39}$$

The quantities E_λ are the energy levels of the atom and the ket $|\mathbf{k},\alpha\rangle$ created by the operator $a_\alpha^\dagger(\mathbf{k})$ represents a photon state of momentum $\mathbf{k}$ and polarization $\mathbf{e}_\alpha(\mathbf{k})$. We now refer to the field expression (4.23) and consider the quantum field

$$\mathbf{A} = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \sum_{\alpha=1}^{2} \mathbf{e}_\alpha(\mathbf{p})\left[a_\alpha(\mathbf{p})e^{-ipx} + a_\alpha^\dagger(\mathbf{p})e^{ipx}\right] \tag{4.40}$$

for which a further approximation can be assumed. Since the length r is of the order of the Bohr radius $a \approx 0.53\times10^{-10}\ m$ and the wavelengths $\lambda \approx 3\times10^{-7}\ m$ are in the visible

or ultra-violet range, we have $pr = 2\pi a/\lambda \approx 10^{-3}$. It is therefore justified to apply the classical **dipole approximation** given by the expansion

$$e^{-i\mathbf{p}\cdot\mathbf{r}} = 1 - i\mathbf{p}\cdot\mathbf{r} + \cdots \tag{4.41}$$

where only the first nonvanishing matrix element will be retained. Thus, with the quantum field (4.40), the "bra" $\langle \mathbf{k}, \alpha| = \langle 0|a_\alpha(\mathbf{k})$ and the commutation relation (4.29), we calculate the expectation value of the interaction Hamiltonian

$$\begin{aligned}
\langle f|H_I|i\rangle &= \frac{e}{m}\langle E_f| \otimes \langle \mathbf{k}, \alpha|\mathbf{A}\cdot\mathbf{p}|E_i\rangle \otimes |0\rangle \\
&\approx \frac{e}{m}\int \frac{d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}'}} \sum_{\alpha'=1}^{2} \langle \mathbf{k}, \alpha|a^\dagger_{\alpha'}(\mathbf{p}')|0\rangle e^{i\omega_{\mathbf{p}'}t}\ \langle E_f|\mathbf{e}_{\alpha'}(\mathbf{p}')\cdot\mathbf{p}|E_i\rangle \\
&= \frac{e}{m}\int \frac{d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}'}} \sum_{\alpha'=1}^{2} \langle 0|a_\alpha(\mathbf{k})a^\dagger_{\alpha'}(\mathbf{p}')|0\rangle e^{i\omega_{\mathbf{p}'}t}\ \langle E_f|\mathbf{e}_{\alpha'}(\mathbf{p}')\cdot\mathbf{p}|E_i\rangle \\
&= \frac{e}{m}\int \frac{d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}'}} \sum_{\alpha'=1}^{2} (2\pi)^3 2\omega_{\mathbf{p}'}\delta(\mathbf{k}-\mathbf{p}')\delta_{\alpha\alpha'} e^{i\omega_{\mathbf{p}'}t}\ \langle E_f|\mathbf{e}_{\alpha'}(\mathbf{p}')\cdot\mathbf{p}|E_i\rangle \\
&= \frac{e}{m}\ e^{i\omega_{\mathbf{k}}t}\langle E_f|\mathbf{e}_\alpha(\mathbf{k})\cdot\mathbf{p}|E_i\rangle.
\end{aligned} \tag{4.42}$$

We go one step further by considering the commutation relation

$$[\mathbf{r}, H_{at}] = \left[\mathbf{r}, \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\right] = \frac{1}{2m}[\mathbf{r}, \mathbf{p}^2] = \frac{i}{m}\,\mathbf{p} \tag{4.43}$$

which allows to write the expectation value

$$\begin{aligned}
\langle E_f|\mathbf{e}_\alpha\cdot\mathbf{p}|E_i\rangle &= -im\langle E_f|\mathbf{e}_\alpha\cdot\Big[\mathbf{r}, H_{at}\Big]|E_i\rangle \\
&= -im\langle E_f|\mathbf{e}_\alpha\cdot\Big(\mathbf{r}H_{at} - H_{at}\mathbf{r}\Big)|E_i\rangle \\
&= -im\ (E_i - E_f)\langle E_f|\mathbf{e}_\alpha\cdot\mathbf{r}|E_i\rangle \\
&= -i\frac{m}{e}\ \omega_{if}\ \langle E_f|\mathbf{e}_\alpha\cdot\boldsymbol{\mu}|E_i\rangle
\end{aligned} \tag{4.44}$$

in terms of the electric dipole moment $\boldsymbol{\mu} = e\mathbf{r}$. The time integration of the probability amplitude (4.36) gives the transition probability per unit time

$$\begin{aligned}
\mathcal{W}_{fi} &= \frac{|c^{(1)}_{fi}|^2}{T} \\
&= \frac{\omega^2_{if}}{T}\Big|\langle E_f|\mathbf{e}_\alpha\cdot\boldsymbol{\mu}|E_i\rangle\Big|^2\ \frac{4\sin^2[(\omega_{\mathbf{k}}-\omega_{if})T/2]}{(\omega_{\mathbf{k}}-\omega_{if})^2} \\
&= \frac{\omega^2_{if}}{T}\Big|\langle E_f|\mathbf{e}_\alpha\cdot\boldsymbol{\mu}|E_i\rangle\Big|^2\ 2\pi T\delta(\omega_{\mathbf{k}}-\omega_{if})
\end{aligned} \tag{4.45}$$

where the last equality is valid for T large. It remains to average over all $\mathbf{k}$ directions and all polarizations $\mathbf{e}_n(\mathbf{k})$ $n = 1, 2$. For the integration over d^3k, we use the Lorentz-invariant measure and define the integration coordinates such that $k_z \parallel \boldsymbol{\mu}$ as shown in Figure 4.3. The polarization vector $\mathbf{e}_1$, which is orthogonal to $\mathbf{k}$, can be placed in the $(\boldsymbol{\mu}, \mathbf{k})$-plane. Therefore the vector $\mathbf{e}_2$, orthogonal to $\mathbf{k}$ as well, is also orthogonal to the dipole vector $\boldsymbol{\mu}$

and disappears from the formula. Then, with the dispersion relation $k = \omega_{\mathbf{k}}$ and the solid angle element $d\Omega_k = \sin\vartheta_k d\vartheta_k d\varphi_k$, the integration gives

$$\begin{aligned}
\mathcal{W}_{fi}^{spon} &= 2\pi\ \omega_{if}^2 \int \frac{\omega_{\mathbf{k}}^2 d\omega_{\mathbf{k}} d\Omega_k}{(2\pi)^3 2\omega_{\mathbf{k}}} \delta(\omega_{\mathbf{k}} - \omega_{if}) \Big|\langle E_f|\mathbf{e}_1(\mathbf{k})\cdot\boldsymbol{\mu}|E_i\rangle\Big|^2 \\
&= \pi\ \omega_{if}^2 \int \frac{d\omega_{\mathbf{k}} d\Omega_k}{(2\pi)^3}\ \omega_{\mathbf{k}}\ \delta(\omega_{\mathbf{k}} - \omega_{if}) \Big|\langle E_f|\mu|E_i\rangle\Big|^2 \sin^2\vartheta_k \\
&= \pi \frac{\omega_{if}^3}{(2\pi)^3} \Big|\langle E_f|\mu|E_i\rangle\Big|^2 \frac{8\pi}{3} \\
&= \frac{\Big|\langle E_f|\mu|E_i\rangle\Big|^2}{3\pi}\ \omega_{if}^3\ . \qquad (4.46)
\end{aligned}$$

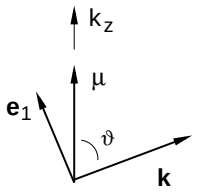

Figure 4.3: Polarization vector $\mathbf{e}_1$ in the $(\boldsymbol{\mu}, \mathbf{k})$-plane

We see that the transition probability $\mathcal{W}_{fi}^{spon}$ is proportional to the third power of the frequency and is non-zero even if the given initial state $|i\rangle = |E_i\rangle \otimes |0\rangle$ is empty of photons. This phenomenon is called **spontaneous emission** and cannot be explained without the quantized field $\mathbf{A}(x)$. This is the first simple physical example that shows the necessity of quantum field theory. The return to SI units is quite simple if we know that the squared electron charge must be replaced by the fine structure constant

$$\frac{e^2}{4\pi} \quad \longrightarrow \quad \alpha = \frac{e^2}{4\pi\epsilon_0}\ \frac{1}{\hbar c}$$

and if we divide the expression (4.46) by c^2. We thus obtain the transition probability per second

$$\mathcal{W}_{fi}^{spon} = \frac{\Big|\langle E_f|\mu|E_i\rangle\Big|^2}{3\pi\epsilon_0\hbar c^3}\ \omega_{if}^3\ . \qquad (4.47)$$

Chapter 5

The Fermion Field

The fermion field is defined by the relativistic equation for a free electron of mass m. What is the form of this equation ? The Klein-Gordon equation (3.1) cannot be used because its probability density ρ is not positive definite. This can be seen by multiplying Eq. (3.1) by ϕ^*, its complex conjugate by ϕ and by performing the difference between the two. We obtain the continuity equation $\partial\rho/\partial t + \nabla \cdot \mathbf{j} = 0$, where

$$\rho = \frac{\hbar}{2im}\left[\phi\partial_t\phi^* - \phi^*\partial_t\phi\right] \qquad \mathbf{j} = \frac{\hbar}{2im}\left[\phi^*\nabla\phi - \phi\nabla\phi^*\right] \ . \tag{5.1}$$

Moreover, the Klein-Gordon equation, as second order in time differential equation, is not suitable for describing a one-particle system with only one initial condition $\phi(\mathbf{r}, t_0)$. It can be transformed into a system of two first order in time differential equations and used, for instance, for describing the evolution of pairs of particles like charged π-mesons.

5.1 Dirac Equation and Dirac Plane Waves

In order to implement a relativistic equation for the electron, we follow Dirac and pick up some guiding principles :

The probability amplitude $\psi(x)$ must :

- have a many-component strucure describing also the spin,
- contain a complete information at t_0, therefore obey a first-order in time equation,
- satisfy the superposition principle, therefore obey a linear homogeneous equation.

The relativistic equation must :

- be linear, homogeneous, Lorentz covariant and, owing to covariance, contain only first-order derivatives with respect to time and space,
- preserve energy-momentum conservation *i.e.* imply the Klein-Gordon equation,
- be invariant by local gauge transformation

$$\psi'(x) = e^{-\frac{ie_0}{\hbar}\chi(x)}\ \psi(x) \ . \tag{5.2}$$

The most general equation satisfying the above-given conditions, excepted for the gauge invariance, can be written under the form

$$\partial_t \psi + \boldsymbol{\alpha} \cdot \nabla \psi + iK\beta\psi = 0 \qquad K \in \mathbb{R} \tag{5.3}$$

where the matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β must be Hermitian. In order to satisfy the relativistic energy-momentum relation, Eq. (5.3) must leads to the Klein-Gordon equation (3.1). This can be achieved (homework) by applying the operator $(-\partial_t + \boldsymbol{\alpha} \cdot \nabla + iK\beta)$ on Eq. (5.3), by setting $K = m$, and by requiring that the matrices $\boldsymbol{\alpha}$, β obey the relations

$$\beta^2 = I \tag{5.4}$$

$$\alpha_j\beta + \beta\alpha_j = 0 \tag{5.5}$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} I \qquad j,k = 1,2,3 \tag{5.6}$$

where I is the unit matrix. From the relation $\beta^2 = I$ and $\alpha_j^2 = I$, we deduce that the matrices $\boldsymbol{\alpha}$, β are also unitary[1] and therefore have eigenvalues ± 1. The dimension N of these matrices is fixed by the relations (5.4), (5.5) and the calculation of the trace[2]

$$Tr(\alpha_j) = Tr(\beta^2\alpha_j) = Tr(\beta\alpha_j\beta) = -Tr(\alpha_j\beta^2) = -Tr(\alpha_j) = 0 \ .$$

Thus, the matrices β, α_j of eigenvalues ± 1 and null trace have an even N. The value $N = 2$ is excluded, since there exists only three 2×2 independent Hermitian matrices such that $AB + BA = 0$. These are the **Pauli matrices**

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ . \tag{5.7}$$

We then take the next even number $N = 4$ and choose the four matrices

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \qquad \boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \tag{5.8}$$

which satisfy the relations (5.4), (5.5) and (5.6). The probability amplitude can be written

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \ . \tag{5.9}$$

The explicit covariant form of the Dirac equation follows from the definition of the four vector

$$(\gamma^\mu) = (\beta, \beta\boldsymbol{\alpha}) \tag{5.10}$$

which has the property $\gamma_\mu^\dagger = \gamma^\mu$ and allows to put the relations (5.4) to (5.6) together

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \ . \tag{5.11}$$

[1] Unitary matrix : $UU^\dagger = I = U^\dagger U$

[2] The trace of a product of square matrices A, B, C has the cyclic property :
$Tr(ABC) = Tr(CAB) = Tr(BCA)$.

Then, the **Dirac equation** for the free electron takes the covariant form[3]

$$[i\gamma^\mu \partial_\mu - m]\,\psi(x) = 0\ . \tag{5.12}$$

With the notation $\not{\partial} = \gamma^\mu \partial_\mu$, we can also write

$$[i\not{\partial} - m]\,\psi(x) = 0\ . \tag{5.13}$$

Moreover, one should show (see textbooks) that the Dirac equation is relativistic invariant *i.e* there exist an invertible matrix $S(\Lambda)$, depending on the Lorentz matrix Λ, and such that

$$\psi'(x') = S(\Lambda)\psi(x) \quad \text{and} \quad S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \tag{5.14}$$

The four-component quantity $\psi(x)$ is called a **Dirac spinor**. If we now left-multiply Eq. (5.13) by γ^0 and take the complex conjugate transpose, we obtain the adjoint equation

$$i\partial_\mu \overline{\psi}(x)\gamma^\mu + m\overline{\psi}(x) = 0 \tag{5.15}$$

where the **adjoint** of $\psi(x)$ is defined as

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0. \tag{5.16}$$

The adjoint equation can also be written

$$\overline{\psi}(x)\left[i\overleftarrow{\not{\partial}} + m\right] = 0\ , \tag{5.17}$$

and the arrow means that the derivative acts on the function to the left. This notation takes into account of the fact that the matrices ψ and γ^μ do not commute. By combining the Dirac equation and its adjoint, we can derive in the usual way the continuity equation

$$\partial_\mu j^\mu = 0 \tag{5.18}$$

where we have defined the **probability current**

$$j^\mu = \overline{\psi}\gamma^\mu\psi. \tag{5.19}$$

The Dirac equation (5.12) can be derived from the Hermitian Lagrangian density

$$\mathcal{L} = \frac{i}{2}\left[\overline{\psi}(x)\not{\partial}\psi(x) - \overline{\psi}(x)\overleftarrow{\not{\partial}}\,\psi(x)\right] - m\overline{\psi}(x)\psi(x). \tag{5.20}$$

For the verification, we consider the field $\overline{\psi}(x)$ and $\psi(x)$ as independent variables and apply (homework) the Euler-Lagrange equations with respect to the components ψ_α. From the definition of the conjugate momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{i}{2}\psi^\dagger \qquad \pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = -\frac{i}{2}\psi, \tag{5.21}$$

we can deduce (homework) the Hamiltonian density

$$\mathcal{H} = \frac{i}{2}\left[\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger\ \psi\right]. \tag{5.22}$$

[3]In standard units, the mass m must be replaced by the quantity $mc/\hbar$.

As for the scalar field and for the electromagnetic field, we seek solutions of the free-electron Dirac equation (5.12). With the covariant Fourier transform

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p) e^{-ipx}, \tag{5.23}$$

we arrive at the eigenvalue equation

$$(\not{p} - m)\tilde{\psi}(p) = 0 \tag{5.24}$$

where $\not{p} = \gamma^\mu p_\mu$. This equation can be explicitly written

$$(p_0 - \boldsymbol{\alpha} \cdot \mathbf{p} - \beta m)\tilde{\psi}(p) = 0 \tag{5.25}$$

and the matrix form (5.8) of $\boldsymbol{\alpha}$ and β shows that $\tilde{\psi}$ can be separated into two two-component spinors

$$\tilde{\psi} \sim \begin{bmatrix} \varphi \\ \chi \end{bmatrix} . \tag{5.26}$$

Then, the eigenvalue equation (5.25) takes the two-equation structure

$$\begin{bmatrix} p_0 - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ -\boldsymbol{\sigma} \cdot \mathbf{p} & p_0 + m \end{bmatrix} \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = 0 \tag{5.27}$$

which has a non-trivial solution if its determinant is equal to zero. The calculation of the determinant uses the property $(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2$ of the Pauli matrices[4] and yields the two energy eigenvalues

$$p_0^{\pm} = \pm\sqrt{m^2 + \mathbf{p}^2} \equiv \pm\epsilon_{\mathbf{p}} \tag{5.28}$$

that show a double degeneracy. The corresponding eigenspinors are determined by writing Eq. (5.27) in two coupled equations

$$\varphi = \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{(p_0 - m)}\, \chi \qquad\qquad \chi = \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{(p_0 + m)}\, \varphi \ . \tag{5.29}$$

Then, for $|\mathbf{p}| \ll m$, we see that the spinors behave as

$$\tilde{\psi} \to \lim_{p_0^+ \to m} \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \lim_{p_0^+ \to m} \begin{bmatrix} \varphi \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{(p_0^+ + m)}\, \varphi \end{bmatrix} \sim \begin{bmatrix} \varphi \\ 0 \end{bmatrix}$$

$$\tilde{\psi} \to \lim_{p_0^- \to -m} \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \lim_{p_0^- \to -m} \begin{bmatrix} \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{(p_0^- - m)}\, \chi \\ \chi \end{bmatrix} \sim \begin{bmatrix} 0 \\ \chi \end{bmatrix}$$

and show a correct non-relativistic correspondence with respect to positive and negative energies. We thus introduce the up ($s = 1$) and down ($s = 2$) spin states

$$\varphi_1 = \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \varphi_2 = \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.30}$$

[4]Properties of the Pauli matrices : $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, \quad \sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2$.

and infer, for the eigenvectors u_s and v_s relative to $p_0^{\pm} = \pm\epsilon_{\mathbf{p}}$, the following expressions

$$u_s(\mathbf{p}) \sim \begin{bmatrix} \varphi_s \\ \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{(\epsilon_{\mathbf{p}}+m)}\ \varphi_s \end{bmatrix} \qquad v_s(-\mathbf{p}) \sim \begin{bmatrix} -\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{(\epsilon_{\mathbf{p}}+m)}\ \chi_s \\ \chi_s \end{bmatrix} \qquad s = 1, 2\ . \quad (5.31)$$

The reason for the definition of v_s with the argument $-\mathbf{p}$ will become clear below. From Eq. (5.24) and (5.25), we see that we can also write the two equations

$$(\not{p} - m)\ u_s(\mathbf{p}) = 0 \quad (5.32)$$
$$(\not{p} + m)\ v_s(\mathbf{p}) = 0 \quad (5.33)$$

where $\not{p} = (\gamma^0\epsilon_{\mathbf{p}} - \boldsymbol{\gamma}\cdot\mathbf{p})$. With the following chosen normalization condition

$$u_s^{\dagger}(\mathbf{p})u_{s'}(\mathbf{p}) = 2\epsilon_{\mathbf{p}}\delta_{ss'} = v_s^{\dagger}(\mathbf{p})v_{s'}(\mathbf{p}) \qquad s = 1, 2\ , \quad (5.34)$$

we obtain (homework) the positive-energy and negative-energy spinors which are called **Dirac plane waves**

$$u_s(\mathbf{p})\ e^{-ipx} = \frac{1}{\sqrt{\epsilon_{\mathbf{p}} + m}} \begin{bmatrix} (\epsilon_{\mathbf{p}} + m)\ \varphi_s \\ \boldsymbol{\sigma}\cdot\mathbf{p}\ \varphi_s \end{bmatrix} e^{-ipx} \quad (5.35)$$

$$v_s(\mathbf{p})\ e^{ipx} = \frac{1}{\sqrt{\epsilon_{\mathbf{p}} + m}} \begin{bmatrix} \boldsymbol{\sigma}\cdot\mathbf{p}\ \chi_s \\ (\epsilon_{\mathbf{p}} + m)\ \chi_s \end{bmatrix} e^{ipx} \qquad s = 1, 2\ . \quad (5.36)$$

In the second formula we have changed the sign of $\mathbf{p}$ and thus recovered a covariant exponent $px = (\epsilon_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{r})$. We also verify (homework) the orthogonality relations

$$\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = 2m\delta_{s,s'} \quad (5.37)$$
$$\bar{v}_s(\mathbf{p})v_{s'}(\mathbf{p}) = -2m\delta_{s,s'} \quad (5.38)$$
$$\bar{u}_s(\mathbf{p})v_{s'}(\mathbf{p}) = 0 = u_s^{\dagger}(\mathbf{p})v_{s'}(-\mathbf{p}) \quad (5.39)$$

where $\bar{u} = u^{\dagger}\gamma^0$. It is straightforward but quite long to check (homework) that the closure relations are given by

$$\sum_{s=1}^{2} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = (\not{p} + m) \quad (5.40)$$
$$\sum_{s=1}^{2} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = (\not{p} - m). \quad (5.41)$$

Finally, the Dirac plane waves

$$\psi_{u_s}(x) = \frac{1}{(2\pi)^{3/2}\sqrt{2\epsilon_{\mathbf{p}}}}\ u_s(\mathbf{p})\ e^{-ipx} \qquad \psi_{v_s}(x) = \frac{1}{(2\pi)^{3/2}\sqrt{2\epsilon_{\mathbf{p}}}}\ v_s(\mathbf{p})\ e^{ipx} \quad (5.42)$$

written with the usual normalization, satify the orthonormality relations

$$\int d^3r\ \psi_{u_s}^{\dagger}(x)\psi_{u_{s'}}(x) = \delta_{ss'}\delta(\mathbf{p} - \mathbf{p}') = \int d^3r\ \psi_{v_s}^{\dagger}(x)\psi_{v_{s'}}(x). \quad (5.43)$$

We can now express the solution of Eq. (5.12) as a three-dimensional integral

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \tilde{\psi}(\mathbf{p}) e^{-ipx} \tag{5.44}$$

which contains the Lorentz-invariant measure. The fonction $\tilde{\psi}(\mathbf{p})$ is given by a linear combination of the basis spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ $s = 1, 2$, and leads to the general formulation of the spinor field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \sum_{s=1}^{2} \Big[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ipx} + d_s^*(\mathbf{p}) v_s(\mathbf{p}) e^{ipx} \Big] \tag{5.45}$$

for any complex number $b_s(\mathbf{p})$ and $d_s^*(\mathbf{p})$. For convenient reasons, we write the second Fourier coefficient as complex conjugate.

At this point, we must give a quantum interpretation to the negative-energy Dirac plane waves. The existence of free electron quantum states with negative energy allows to radiate photons with infinite energy. Indeed, an electron with positive energy ϵ_{p_1} can decay into one with negative energy $-\epsilon_{p_2}$, radiating a photon with total energy $\epsilon_{p_1} + \epsilon_{p_2}$. Going down in energy, the radiation of the photon could even be infinite. Dirac circumvents the problem by supposing that all the negative-energy states are already filled with electrons and introduced, in this way, a new ground state called the **Dirac sea**. The Pauli exclusion principle prevents a positive-energy electron to occupy a filled negative-energy state. However, one of these negative-energy electrons can still be excited into a positive-energy state, whereupon it becomes a real electron, leaving behind a hole. A **hole** is an absence of negative electric charge or a positive charge. This reasoning leads to the prediction of the existence of a positively charged spin 1/2 particle, the **positron**, discovered in cosmic rays, by C. D. Anderson in 1932.

5.2 Relativistic Electron in an Electromagnetic Field

The Dirac equation for an electron of charge $-e$ in an electromagnetic field can be implemented from the **gauge principle** which requires that the equation must be invariant by local gauge transformation

$$\psi'(x) = e^{ie\chi(x)}\ \psi(x). \tag{5.46}$$

This relation inserted into Eq. (5.12) gives the expression

$$\left[i\gamma^\mu(\partial_\mu + ie\partial_\mu\chi) - m \right] \psi'(x) = 0 \tag{5.47}$$

which is clearly not invariant under the gauge tranformation (5.46). However, we remark that the new term $ie\partial_\mu\chi$ can be balanced by a **gauge field** $A_\mu(x)$ obeying the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\chi\ . \tag{5.48}$$

It is then easy to verify that the equation

$$\Big[i\gamma^\mu(\partial_\mu - ieA_\mu) - m \Big] \psi(x) = 0 \tag{5.49}$$

is gauge invariant *i.e.* keeps the same form after the gauge transformations (5.46) and (5.48). The quantity A_μ can be identified with the electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ which is connected to $\mathbf{E}$ and $\mathbf{B}$ by the relations

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{5.50}$$

The definition of the **covariant derivative**[5]

$$D_\mu = \partial_\mu - ieA_\mu \tag{5.51}$$

and the notation $\not{D} = \gamma^\mu D_\mu$ allow to write the Dirac equation for an electron in an electromagnetic field in a nice and compact expression

$$[i\not{D} - m]\,\psi(x) = 0 \ . \tag{5.52}$$

The one-particle relativistic quantum mechanics can be illustrated by the two examples given below.

5.2.1 Relativistic Hydrogen Atom

The calculation of the energy levels of the relativistic hydrogen atom can be done as usual by considering a relativistic electron in the Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{r} \ . \tag{5.53}$$

From the Dirac equation (5.49), we see that the energy observable is given by the time-independant Dirac Hamiltonian

$$H = c\left(\boldsymbol{\alpha}\cdot\frac{\hbar}{i}\nabla\right) + mc^2\beta + V(r) \tag{5.54}$$

whose eigenstates must belong to the Hilbert space of square integrable functions. The energy E and the function $u(\mathbf{r})$, depending also on the spin degree of freedom, are characterized by the stationary wave function

$$\psi(\mathbf{r}, t) = u(\mathbf{r})\ e^{-\frac{i}{\hbar}Et} \tag{5.55}$$

and the corresponding Hamiltonian eigenvalue equation

$$H\ u(\mathbf{r}) = E\ u(\mathbf{r}) \ . \tag{5.56}$$

We must solve this equation by using the standard methods of quantum mechanics. In spherical coordinates, the Hamiltonian (5.54) has a radial dependence and also includes the angular momentum operator

$$\mathbf{L} = \mathbf{r} \times \frac{\hbar}{i}\nabla$$

[5]In SI units we have $\psi'(x) = e^{ie/\hbar)\chi(x)}\ \psi(x)$ and $D_\mu = \partial_\mu - \frac{ie}{\hbar}A_\mu$. Moreover, we can also remark that the gauge field $A_\mu(x)$ appearing in the covariant derivative has a geometrical meaning similar to the Christoffel symbols occuring in Riemannian geometry.

and the Dirac **spin operator**

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} = \frac{\hbar}{2}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \boldsymbol{\sigma} \ . \tag{5.57}$$

This last operator can be related to the matrices $\boldsymbol{\alpha}$ by the expression

$$\boldsymbol{\alpha} = \gamma^5 \mathbf{\Sigma} \qquad\qquad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \ .$$

How to display the radial and the spin-angular dependences of H ? This tricky task can be achieved by considering the momentum operator $\frac{\hbar}{i}\nabla$ in the following two identities[6]

$$\mathbf{e}_r \times (\mathbf{e}_r \times \frac{\hbar}{i}\ \nabla) = \frac{1}{r}\ [\mathbf{e}_r \times \mathbf{L}]$$
$$\mathbf{e}_r \times (\mathbf{e}_r \times \frac{\hbar}{i}\nabla) = \mathbf{e}_r(\mathbf{e}_r \cdot \frac{\hbar}{i}\nabla) - \frac{\hbar}{i}\nabla$$

which lead to the expression

$$\boldsymbol{\alpha} \cdot \frac{\hbar}{i}\nabla = \boldsymbol{\alpha} \cdot \mathbf{e}_r(\mathbf{e}_r \cdot \frac{\hbar}{i}\nabla) - \boldsymbol{\alpha} \cdot \frac{1}{r}[\mathbf{e}_r \times \mathbf{L}]. \tag{5.58}$$

Then, with the definitions $\alpha_r = \boldsymbol{\alpha} \cdot \mathbf{e}_r$, $\partial/\partial r = \mathbf{e}_r \cdot \nabla$ and by using the properties of the Pauli matrices[7], Eq. (5.58) can also be expressed (homework) as

$$\boldsymbol{\alpha} \cdot \frac{\hbar}{i}\nabla = \alpha_r \left[\frac{\hbar}{i}\frac{\partial}{\partial r} + \frac{2i}{\hbar}\frac{(\mathbf{L}\cdot\mathbf{S})}{r}\right] \ . \tag{5.59}$$

We can then write the Dirac Hamiltonian (5.54) in a convenient form separated into radial and spin-angular terms

$$H = c\ \alpha_r \left[\frac{\hbar}{i}\frac{\partial}{\partial r} + \frac{2i}{\hbar}\frac{(\mathbf{L}\cdot\mathbf{S})}{r}\right] + mc^2\beta + V(r) \ . \tag{5.60}$$

The operator $\mathbf{L}\cdot\mathbf{S}$ is called **spin-orbit operator**. It can be determined by considering the total angular momentum[8] operator

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{5.61}$$

whose square gives the expression

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}\left[\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right] \ . \tag{5.62}$$

It is now straightforward to check (do it !) that the set of operators

$$\left\{H, \mathbf{J}^2, J_z, \mathbf{S}^2\right\} \tag{5.63}$$

[6]The vector $\mathbf{e}_r$ is the radial unit vector. The first identity simply results from the definition of the angular momentum. The second is the well-known relation between vector en scalar products.

[7]The properties of the Pauli matrices given in the footnote[4] are obviously also satisfied by the Dirac spin operator $\mathbf{\Sigma}$. Then, it is easy to verify that $\alpha_1\Sigma_2 = i\alpha_3,\ \alpha_2\Sigma_3 = i\alpha_1,\ \alpha_3\Sigma_1 = i\alpha_2$.

[8]The correct way to write this operator is $\mathbf{J} = \mathbf{L}\otimes I + I \otimes \mathbf{S}$ where I is the identity operator.

is a complete set of commuting observables, which allows to factorize the eigenvectors of Eq. (5.56) into a tensor product of four vectors. Let us first consider the spin-angular operators. From the eigenvectors $Y_{lm_l}(\vartheta,\varphi)$ (spherical harmonics) of the angular momentum obserbvables $\mathbf{L}^2, L_z$ and from the two-component[9] eigenspinors χ_{m_s} of the spin obserbvables $\mathbf{S}^2, S_z$, the eigenstates of the commuting operators $\mathbf{J}^2, J_z, \mathbf{L}^2, \mathbf{S}^2$ can be written in terms of Clebsch-Gordan coefficients[10] as a linear combination

$$\mathcal{Y}_{jm}(l|\vartheta,\varphi) = \sum_{m_l,m_s} \langle l, m_l, \tfrac{1}{2}, m_s | j, m\rangle \; Y_{lm_l}(\vartheta,\varphi)\chi_{m_s} \tag{5.64}$$

where the quantum numbers take the values

$$\begin{aligned} & l = 0, 1, 2, \cdots \qquad\qquad -l \le m_l \le l \qquad m_s = \pm\frac{1}{2} \\ & |l - \frac{1}{2}| \le j \le l + \frac{1}{2} \qquad\quad m = m_l + m_s \qquad\qquad . \end{aligned} \tag{5.65}$$

With the eigenvalue equations

$$\begin{aligned} \mathbf{J}^2 \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) &= \hbar^2 j(j+1) \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) \\ J_z \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) &= \hbar \; m \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) \\ \mathbf{L}^2 \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) &= \hbar^2 (j \mp \tfrac{1}{2})(j \mp \tfrac{1}{2} + 1) \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) \\ \mathbf{S}^2 \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) &= \frac{3}{4}\hbar^2 \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) \end{aligned}$$

and from the definition (5.62) of the spin-orbit operator, we obtain

$$(\mathbf{L}\cdot\mathbf{S}) \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) = -\frac{\hbar^2}{2}\left[1 + \kappa\right] \; \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) \tag{5.66}$$

$$\kappa = \begin{cases} -(j + \frac{1}{2}) & j = l + \frac{1}{2} \\ +(j + \frac{1}{2}) & j = l - \frac{1}{2} \end{cases} . \tag{5.67}$$

The Dirac Hamiltonian (5.54) commute (check it !) with the **Dirac parity operator**[11]. Hence, we can construct even and odd eigenstates common to both operators and write

[9] The form of the spin operator (5.57) allows to consider two-component spinors.

[10] These Clebsch-Gordan coefficients $\langle l, m_l, \frac{1}{2}, m_s | j, m\rangle$ are the coefficients of the linear transformation which allows to pass, from the tensor product eigenbasis $Y_{lm_l}(\vartheta,\varphi)\chi_{m_s}$ common to the obserbvables $\mathbf{L}^2, L_z$, $\mathbf{S}^2, S_z$, to the more appropriate eigenbasis $\mathcal{Y}_{jm}(l|\vartheta,\varphi)$ common to the observables $\mathbf{J}^2, J_z$, $\mathbf{L}^2, \mathbf{S}^2$.

[11] The parity operator is defined by the transformation $P : \mathbf{r} \longmapsto -\mathbf{r}$. For the Dirac spinors, the parity operator is defined by a general transformation such as (5.14), where the Lorentz matrix is replaced by the spatial reflection

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

Then, up to an unimportant phase factor, we can show that the Dirac parity operator is given by

$$u'(r') \equiv \beta u(\mathbf{r}) = \pm u(-\mathbf{r}) \; .$$

Furthermore, we know that for a fixed j, the eigenfunctions $\mathcal{Y}_{jm}(j \mp \frac{1}{2}|\vartheta,\varphi)$ have opposite parities

$$\mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\pi - \vartheta, \varphi + \pi) = \begin{cases} (-1)^{(j-\frac{1}{2})} \; \mathcal{Y}_{jm}(j - \frac{1}{2}|\vartheta,\varphi) \\ -(-1)^{(j-\frac{1}{2})} \; \mathcal{Y}_{jm}(j + \frac{1}{2}|\vartheta,\varphi) \end{cases}$$

since the parity of the spherical harmonics is given by $(-1)^l$ with $l = j \mp \frac{1}{2}$.

the eigenfunctions in the general form

$$u(\mathbf{r}) = \frac{1}{r}\left[\begin{array}{l} i\ F(r)\ \ \mathcal{Y}_{jm}(j \mp \frac{1}{2}|\vartheta,\varphi) \\ -\ G(r)\ \mathcal{Y}_{jm}(j \pm \frac{1}{2}|\vartheta,\varphi) \end{array}\right] \tag{5.68}$$

where the factors i and $1/r$ are introduced for convenient reasons. It is now quite an easy task to apply the Hamiltonian (5.60) on these functions. Besides the well-defined effect of the spin-orbit operator (5.66) and of the derivative ∂_r, one must still specify the action of the spin operator

$$\alpha_r = \gamma^5\ \boldsymbol{\Sigma}\cdot\hat{\mathbf{r}} = \left[\begin{array}{cc} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{r}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{r}} & 0 \end{array}\right] \tag{5.69}$$

whose pseudoscalar components $\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$ have the property

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\ \mathcal{Y}_{jm}(j \mp \tfrac{1}{2}|\vartheta,\varphi) = -\mathcal{Y}_{jm}(j \pm \tfrac{1}{2}|\vartheta,\varphi) \tag{5.70}$$

which can be derived (homework) from the commutator $\left[J_k\,,\,\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right] = 0,\ k = 1.2.3$. With these results, the Hamiltonian (5.60) is applied on the eigenfunctions (5.68) and leads (do it !) to the following system of two differential equations

$$\left[\frac{1}{\hbar c}(E - mc^2) + \frac{Z\alpha}{r}\right] F(r) \ = \ -\frac{dG(r)}{dr} + \frac{\kappa}{r}G(r) \tag{5.71}$$

$$\left[\frac{1}{\hbar c}(E + mc^2) + \frac{Z\alpha}{r}\right] G(r) \ = \ +\frac{dF(r)}{dr} + \frac{\kappa}{r}F(r) \tag{5.72}$$

where α is the fine structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c} \simeq \frac{1}{137}\ . \tag{5.73}$$

The solution of this system of first order differential equations can be found as usual by a series Ansatz and by the application of the condition $\int |u(\mathbf{r})|^2 d^3r = 1$. The calculations are quite long and give as radial eigenfunctions the half-integer Laguerre polynomials and as eigenvalues the following energy levels

$$E_{n,j} = mc^2\left[1 + \left(\frac{Z\alpha}{n - (j+\frac{1}{2}) + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}}\right)^2\right]^{-1/2} \tag{5.74}$$

$$n = 1, 2, 3, \cdots$$
$$j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots$$

This splitting of the energy levels due to spin and relativistic effects is called the **fine structure splitting**. Another small splitting between ${}^2s_{1/2}$ and ${}^2p_{1/2}$ states, called the **Lamb shift**, is only predicted by quantum field theory.

5.2.2 Integral Equation and Coulomb Scattering

Relativistic quantum mechanics is also well adapted to the description of scattering phenomena. In these cases, Eq. (5.52) can be solved approximately by using a simple and intuitive method proposed by Feynman. Let us first write the Dirac equation in the explicit form

$$(i\partial\!\!\!/ - m)\psi(x) = -e\, A\!\!\!/\psi(x) \tag{5.75}$$

that can be easily transformed into an integral equation. The Green function or **Feynman propagator** $S_F(x-x')$ defined by the differential equation

$$(i\partial\!\!\!/ - m)S_F(x-x') = \delta^{(4)}(x-x') \tag{5.76}$$

allows to write (check it !) the solution of (5.75) as an integral equation

$$\psi(x) = \psi_0(x) - e\int d^4x'\ S_F(x-x')\, A\!\!\!/(x')\psi(x') \tag{5.77}$$

where $\psi_0(x)$ is a solution of the free-electron Dirac equation. Starting from $\psi_0(x)$, Eq. (5.77) can be iterated by successive substitution and gives the solution

$$\begin{aligned}\psi(x) &= \psi_0(x) - e\int d^4x'\ S_F(x-x')A\!\!\!/(x')\psi_0(x') \\ &+ e^2\int d^4x'\int d^4x''\ S_F(x-x')A\!\!\!/(x')S_F(x'-x'')A\!\!\!/(x'')\psi_0(x'') \\ &- e^3\int d^4x'\int d^4x''\int d^4x''\ \ldots\end{aligned} \tag{5.78}$$

Thus, if $S_F(x-x')$ is known, the field $\psi(x)$ can be found at different orders of perturbation. By Fourier transforming Eq. (5.76), we get the algebraic equation

$$(p\!\!\!/ - m)\tilde{S}(p) = 1 \tag{5.79}$$

where the function $\tilde{S}(p)$ is determined by inversion

$$\tilde{S}(p) = (p\!\!\!/ - m)^{-1} = (p\!\!\!/ + m)(p\!\!\!/ + m)^{-1}(p\!\!\!/ - m)^{-1} = (p\!\!\!/ + m)(p\!\!\!/^2 - m^2)^{-1}\ . \tag{5.80}$$

With the property (5.11) of the γ^μ matrices, the expression

$$p\!\!\!/^2 = \gamma^\mu\gamma^\nu p_\mu p_\nu = 2p_\mu p^\mu - \gamma^\nu\gamma^\mu p_\mu p_\nu = 2p^2 - p\!\!\!/^2\ , \tag{5.81}$$

gives $p\!\!\!/^2 = p^2$ and allows to write the Feynman propagator in p-space

$$\tilde{S}(p) = \frac{(p\!\!\!/ + m)}{(p^2 - m^2)}\ . \tag{5.82}$$

Finally, by inserting $\tilde{S}(p)$ into the Fourier transform, we arrive at the integral

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4}\ \frac{(p\!\!\!/ + m))}{(p^2 - m^2 + i\epsilon)}\ e^{-ip(x-x')} \tag{5.83}$$

where, as shown in Figure 5.1, the small parameter ϵ displays the displacement of the poles $p_0 = \pm\epsilon_{\mathbf{p}}$, and indicates the choice of the integration path in the complex p_0-plane.

By means of the Jordan's lemma, this path can be closed by a large semi-circle in the upper half plane if $t - t' < 0$ and in the lower half plane if $t - t' > 0$. The contribution of these semi-circles tends to zero when the radius tends to infinity and shows that the integration on the closed path is equal to the integration along the real axis. Finally, the application of the residue theorem gives the **Feynman propagator** in x-configuration

$$\begin{aligned} S_F(x-x') &= -i\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \Big[\theta(t-t')\; e^{-ip(x-x')}(\not{p}+m) \\ &\qquad -\theta(t'-t)\; e^{ip(x-x')}(\not{p}-m)\Big] \end{aligned} \tag{5.84}$$

where $p = p^\mu = (\epsilon_{\mathbf{p}}, \mathbf{p})$, $\epsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $\theta(t)$ is the step function. With the closure relation (5.41) and the Dirac plane wave (5.42), the propagator takes the form

$$S_F(x-x') = -i\int d^3p \left[\theta(t-t')\sum_{s=1}^{2}\psi_{u_s}(x)\overline{\psi}_{u_s}(x') - \theta(t'-t)\sum_{s=1}^{2}\psi_{v_s}(x)\overline{\psi}_{v_s}(x')\;\right]. \tag{5.85}$$

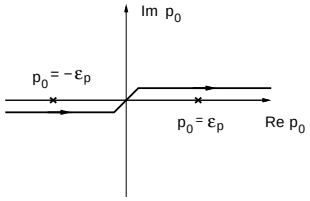

Figure 5.1: Integration path

For $t \to \infty$, we insert $S_F(x-x')$ into Eq. (5.77) and obtain an explicit form of the integral equation

$$\psi(x) = \psi_0(x) + \int d^3p\;\sum_{s=1}^{2}\psi_{u_s}(x)\left[\; ie\int d^4x'\;\overline{\psi}_{u_s}(x')\,\not{A}(x')\;\psi(x')\right] \tag{5.86}$$

that allows, by use of the orthonormality condition (5.43), to express[12] the probability amplitude

$$\begin{aligned} \langle\psi_{u_{s'}}|\psi\rangle &= \langle\psi_{u_{s'}}|\psi_0\rangle + \int d^3r\;\psi^\dagger_{u_{s'}}(x)\int d^3p\;\sum_{s=1}^{2}\psi_{u_s}(x)\left[\; ie\int d^4x'\overline{\psi}_{u_s}(x')\,\not{A}(x')\;\psi(x')\right] \\ &= \langle\psi_{u_{s'}}|\psi_0\rangle + \int d^3p\;\sum_{s=1}^{2}\delta_{ss'}\;\delta(\mathbf{p}-\mathbf{p}')\left[\; ie\int d^4x'\overline{\psi}_{u_s}(x')\,\not{A}(x')\;\psi(x')\right] \\ &= \langle\psi_{u_{s'}}|\psi_0\rangle + ie\int d^4x'\;\overline{\psi}_{u_{s'}}(x')\,\not{A}(x')\;\psi(x'). \end{aligned} \tag{5.87}$$

Scattering phenomena are investigated by cross-section measurements that give essentially the transition probability between free initial and final states. In quantum theory, the **transition probability** from a state $|\psi(t)\rangle$ to a final state $|\psi_f\rangle$ is given by the expression

$$P_{fi} = |\langle\psi_f|\psi(t)\rangle|^2 \tag{5.88}$$

[12]Use the closure relation $\int d^3r\;|\mathbf{r}\rangle\langle\mathbf{r}| = 1$ and obtain $\langle\phi|\psi\rangle = \int d^3r\;\langle\phi|\mathbf{r}\rangle\langle\mathbf{r}|\psi\rangle = \int d^3r\;\phi(\mathbf{r})^*\psi(\mathbf{r})$.

where $|\psi(t)\rangle$ satisfy the initial condition $\lim_{t\to-\infty}|\psi(t)\rangle = |\psi_i\rangle$ and the bracket

$$S_{fi} = \langle\psi_f|\psi(t)\rangle \tag{5.89}$$

is called **scattering amplitude** or **scattering matrix**. The substitution of $\psi(x)$ by $\psi_i(x)$ in (5.87) leads to the first-order approximation

$$S_{fi}^{(1)} = \delta_{fi} + ie\int d^4x\ \bar{\psi}_f(x)\not{A}\psi_i(x) \tag{5.90}$$

that depends only on the free initial and final states and on the electromagnetic field. With the definition of the probability current $j^\mu(x) = \bar{\psi}_f(x)\gamma^\mu\psi_i(x)$, one can also write

$$S_{fi}^{(1)} = \delta_{fi} + ie\int d^4x\ j^\mu(x)A_\mu(x)\ . \tag{5.91}$$

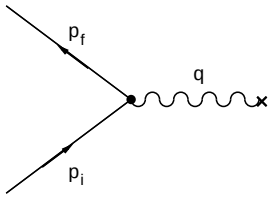

Figure 5.2: Coulomb scattering

As an example of relativistic quantum scattering, let us consider the **Coulomb scattering** of an electron on the potential

$$[A^\mu(x)] = [\Phi, \mathbf{0}] = \left[\frac{Ze}{4\pi}\frac{1}{|\mathbf{r}|}\ , \mathbf{0}\right]\ . \tag{5.92}$$

The integration of (5.90) over dt gives, for $f \neq i$, the expression

$$S_{fi}^{(1)} = -i\frac{Z\alpha}{(2\pi)^2\sqrt{4\epsilon_{\mathbf{p}_f}\epsilon_{\mathbf{p}_i}}}\ \delta(\epsilon_{\mathbf{p}_f} - \epsilon_{\mathbf{p}_i})\int d^3r\ \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{r}\ \bar{u}_f(\mathbf{p}_f)\gamma^0 u_i(\mathbf{p}_i) \tag{5.93}$$

with $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$ and with the fine structure constant[13]

$$\alpha = \frac{e^2}{4\pi}\ .$$

In order to calculate this integral, we must introduce a regularization factor given by the exponential $\exp(-\epsilon r)$ where $\epsilon \to 0$. We obtain (homework)

$$S_{fi}^{(1)} = i\frac{Z\alpha}{(2\pi)^2\sqrt{4\epsilon_{\mathbf{p}_f}\epsilon_{\mathbf{p}_i}}}\ 4\pi\delta(\epsilon_{\mathbf{p}_f} - \epsilon_{\mathbf{p}_i})\frac{1}{|\mathbf{q}|^2}\ \bar{u}_f(\mathbf{p}_f)\gamma^0 u_i(\mathbf{p}_i)\ . \tag{5.94}$$

The $\delta(\epsilon_{\mathbf{p}_f} - \epsilon_{\mathbf{p}_i})$ function expresses the energy conservation. Moreover, we see that it is possible to associate, to the distinct factors of this formula, graphical elements that

[13] In SI units, we have $\alpha = (e^2/4\pi\varepsilon_0)(1/\hbar c)$.

shapes the structure of the **Feynman diagram** of Figure 5.2. More details on these correspondence rules will be given later on. The differential cross-section is provided by the ratio between the probability transition per unit time and the flux of ingoing electrons. It must be multiplied by the number of final momentum states included between $\mathbf{p}_f$ and $\mathbf{p}_f + d\mathbf{p}_f$ and reads

$$d\sigma = \frac{1}{|\mathbf{j}_i|} \frac{|S_{fi}^{(1)}|^2}{T} d^3p_f. \tag{5.95}$$

The calculation (homework) of the flux of incident particles gives

$$\mathbf{j}_i = \bar{\psi}\boldsymbol{\gamma}\psi = \frac{1}{(2\pi)^3} \frac{\mathbf{p}_i}{\epsilon_{\mathbf{p}_i}} \tag{5.96}$$

where the function ψ are the Dirac plane waves (5.42). In order to evaluate the expression (5.95), to integrate over the final energies and to calculate the mean value of the spins, we need the following results :

- the square of the δ function gives

$$|\delta(\omega)|^2 = \left| \int_{-T/2}^{+T/2} \frac{dt}{2\pi} e^{i\omega t} \right|^2 = \frac{1}{\pi^2} \frac{\sin^2(\omega T/2)}{\omega^2} \longrightarrow \frac{T}{2\pi}\delta(\omega) \tag{5.97}$$

- the energy $\epsilon_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$ allows to write $d^3p = |\mathbf{p}|^2 dp d\Omega = |\mathbf{p}|\ \epsilon_{\mathbf{p}} d\epsilon_{\mathbf{p}} d\Omega$

- the enegy-momentum relativistic ratio reads

$$\frac{\mathbf{p}^2}{\epsilon_{\mathbf{p}}^2} = \frac{(\sqrt{1-\beta^2}\ mv)^2}{(\sqrt{1-\beta^2}\ m)^2} = v^2$$

- the spin sum becomes a trace, by use of the closure relation

$$\begin{aligned}
\sum_{i,f} \left| \overline{u}_f \gamma^0 u_i \right|^2 &= \sum_{i,f} (\overline{u}_f \gamma^0 u_i)(\overline{u}_i \gamma^0 u_f) = \sum_f \overline{u}_f \gamma^0 \sum_i u_i \overline{u}_i \gamma^0 u_f \\
&= \sum_f \overline{u}_f \gamma^0 (\not{p}_i + m)\gamma^0 u_f = \sum_{a,b} \sum_f \overline{u}_f^a [\gamma^0 (\not{p}_i + m)\gamma^0]^{ab} u_f^b \\
&= \sum_{a,b} [\gamma^0 (\not{p}_i + m)\gamma^0]^{ab} \sum_f u_f^b \overline{u}_f^a = \sum_{a,b} [\gamma^0 (\not{p}_i + m)\gamma^0]^{ab} [\not{p}_f + m]^{ba} \\
&= Tr\left[\gamma^0 (\not{p}_i + m)\gamma^0 (\not{p}_f + m)\right]
\end{aligned} \tag{5.98}$$

- the trace properties valid for any four-vectors $A^\mu, B^\mu, C^\mu, D^\mu$

$$\begin{aligned}
&1)\ Tr[\text{ odd number of } \gamma^\mu] = 0 \\
&2)\ Tr[\not{A}\not{B}] = 4A^\mu B_\mu = 4AB \\
&3)\ Tr[\not{A}\not{B}\not{C}\not{D}] = 4[(AB)(CD) + (AD)(BC) - (AC)(BD)]\ .
\end{aligned} \tag{5.99}$$

Putting all these results together (homework), we arrive, after a comeback to SI units, at the **Mott cross-section**

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha\hbar c}{2\epsilon_{\mathbf{p}}} \right)^2 \frac{\left[1 - \beta^2 \sin^2(\vartheta/2)\right]}{\beta^4 \sin^4(\vartheta/2)} \tag{5.100}$$

where ϑ is the angle between $\mathbf{p}_i$ and $\mathbf{p}_f$, $\epsilon_{\mathbf{p}}^2 = (mc^2)^2 + (pc)^2$, and $\beta = v/c$. At the non-relativistic limit $\beta \ll 1$, we recover the Rutherford cross-section.

5.3 Quantization of the Fermion Field

As for the other fields, the quantization of the Dirac field considers the Fourier expansion (5.45) and replaces the coefficients $d_s^*(\mathbf{p})$, $b_s(\mathbf{p})$ by creation and annihilation operators $d_s^\dagger(\mathbf{p})$, $b_s(\mathbf{p})$ acting on a Hilbert space. We then arrive at the **fermion field operator**

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \sum_{s=1}^{2} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ipx} \right] \tag{5.101}$$

where $px = \epsilon_p t - \mathbf{p}\cdot\mathbf{r}$ and $\epsilon_p^2 = \mathbf{p}^2 + m^2$. In order to express the Hamiltonian in terms of creation and annihilation operators, from (5.101) we calculate the quantities $\psi^\dagger$, $\partial_t\psi$, $\partial_t\psi^\dagger$ and put them into (5.22). Because of the minus sign in H, the cross terms cancel each other and it remains

$$H = \frac{i^2}{2} \int d^3r \int \frac{d^3p\; d^3p'}{(2\pi)^6 4\epsilon_p \epsilon'_p} \epsilon'_p \sum_{s,s'} \left[-2 b_s^\dagger b_{s'} u_s^\dagger u_{s'}\; e^{i(p-p')x} + 2 d_s d_{s'}^\dagger v_s^\dagger v_{s'}\; e^{-i(p-p')x} \right] \; .$$

The integration over d^3r gives a Dirac $\delta(\mathbf{p}-\mathbf{p}')$ function which absorbs the d^3p' integral. Then, the application of the orthonormality relations

$$u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) = 2\epsilon_p \delta_{ss'} = v_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) \tag{5.102}$$

leads (homework) to the Hamiltonian operator

$$H = \int \frac{d^3p}{(2\pi)^3 2\epsilon_p} \sum_{s=1}^{2} \epsilon_p \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \right] . \tag{5.103}$$

We emphasize the covariant normalization by keeping ϵ_p in the formula. If we define the vacuum state $|0\rangle$ of the Dirac field as the state annihilated by all annihilation operators

$$b_s(\mathbf{p})|0\rangle = 0 \qquad\qquad d_s(\mathbf{p})|0\rangle = 0 \; , \tag{5.104}$$

we could expect, like for the scalar field, an infinite ground state energy removed by the normal ordering (1.62). However, the Hamiltonian (5.103) contains a minus sign and is therefore not positive definite. Nor is the usual normal-ordered Hamiltonian, unless we require that the operators anti-commute *i.e.* obey the new normal product

$$:d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}): \; = -d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \; . \tag{5.105}$$

It turns out that the quantization of the fermion field needs an **anticommutator** symbolically written

$$\{A, B\} = AB + BA \; . \tag{5.106}$$

The requirement of anticommutation is obviously also valid for $b_s(\mathbf{p})$ operators and will be strongly supported by the antisymmetry of the fermionic quantum states, as we will see below. Then, the normal-ordered Hamiltonian

$$:H: \; = \int \frac{d^3p}{(2\pi)^3 2\epsilon_p} \sum_{s=1}^{2} \epsilon_p \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \right] \tag{5.107}$$

possesses the zero ground energy

$$:H:|0\rangle = 0 \; . \tag{5.108}$$

The replacement of commutators by anticommutators leads to the new quantization rules for fermion fields

$$\left\{\psi_a(\mathbf{r},t),\psi_b^\dagger(\mathbf{r}',t)\right\} = i\delta(\mathbf{r}-\mathbf{r}')\delta_{ab} \qquad a,b=1,\cdots,4 \tag{5.109}$$

$$\{\psi_a(\mathbf{r},t),\psi_b(\mathbf{r}',t)\} = 0 = \{\pi_a(\mathbf{r},t),\pi_b(\mathbf{r}',t)\}\ . \tag{5.110}$$

The calculation (homework) of these relations provides the **anticommutation relations** for fermionic creation and annihilation operators

$$\left\{b_s(\mathbf{p})\ ,\ b_{s'}^\dagger(\mathbf{p}')\right\} = (2\pi)^3 2\epsilon_{\mathbf{p}}\delta_{ss'}\delta(\mathbf{p}-\mathbf{p}') \tag{5.111}$$

$$\{b_s(\mathbf{p})\ ,\ b_{s'}(\mathbf{p}')\} = 0 = \left\{b_s^\dagger(\mathbf{p})\ ,\ b_{s'}^\dagger(\mathbf{p}')\right\} \tag{5.112}$$

and also

$$\left\{d_s(\mathbf{p})\ ,\ d_{s'}^\dagger(\mathbf{p}')\right\} = (2\pi)^3 2\epsilon_{\mathbf{p}}\delta_{ss'}\delta(\mathbf{p}-\mathbf{p}') \tag{5.113}$$

$$\{d_s(\mathbf{p})\ ,\ d_{s'}(\mathbf{p}')\} = 0 = \left\{d_s^\dagger(\mathbf{p})\ ,\ d_{s'}^\dagger(\mathbf{p}')\right\}\ . \tag{5.114}$$

All couples of operators $b_s(\mathbf{p})$, $d_s(\mathbf{p})$ are obviously anti-commuting. In this theory, four kinds of one-particle states can be created by the four operators $b_s^\dagger(\mathbf{p})$, $d_s^\dagger(\mathbf{p})$ $s=1,2$. They are written as follows

$$|\mathbf{p},s\rangle_+ = b_s^\dagger(\mathbf{p})|0\rangle \tag{5.115}$$

$$|\mathbf{p},s\rangle_- = d_s^\dagger(\mathbf{p})|0\rangle \qquad s=1,2\ . \tag{5.116}$$

Many-particle states are built as usual from the tensor product of single-particle states. For example, a two-particle state reads

$$|\mathbf{p}_1,s_1\rangle_+ \otimes |\mathbf{p}_2,s_2\rangle_+ = b_{s_1}^\dagger(\mathbf{p}_1)b_{s_2}^\dagger(\mathbf{p}_2)|0\rangle\otimes|0\rangle \tag{5.117}$$

where it is implicitly meant that the first operator acts on the first state and the second operator on the second state. Moreover, the interchange of particles implies the anticommution of the creation operators and assigns to the new state a relative minus sign

$$\begin{aligned} |\mathbf{p}_1,s_1\rangle_+ \otimes |\mathbf{p}_2,s_2\rangle_+ &= b_{s_1}^\dagger(\mathbf{p}_1)b_{s_2}^\dagger(\mathbf{p}_2)|0\rangle\otimes|0\rangle \\ &= -b_{s_2}^\dagger(\mathbf{p}_2)b_{s_1}^\dagger(\mathbf{p}_1)|0\rangle\otimes|0\rangle \\ &= -|\mathbf{p}_2,s_2\rangle_+ \otimes |\mathbf{p}_1,s_1\rangle_+\ . \end{aligned} \tag{5.118}$$

We deduce that the states of the Dirac field are **antisymmetric** under particle interchange

$$|\mathbf{p}_1,s_1\rangle_+ \otimes |\mathbf{p}_2,s_2\rangle_+ = -|\mathbf{p}_2,s_2\rangle_+ \otimes |\mathbf{p}_1,s_1\rangle_+\ . \tag{5.119}$$

We call these particles **fermions**[14]. Furthermore, if we suppose both momenta and both spins equal to $\mathbf{p}$ and s respectively, we find

$$|\mathbf{p},s\rangle_+ \otimes |\mathbf{p},s\rangle_+ = -|\mathbf{p},s\rangle_+ \otimes |\mathbf{p},s\rangle_+ = 0 \tag{5.120}$$

[14]The antisymmetry comes here in a natural way from the definition of the Dirac Hamiltonian and from the quantization, whereas in many-particle quantum mechanics, it has to be taken for granted.

and infer that two fermions can never be in the same state. This is precisely the **Pauli exclusion principle**. The occupation number for fermions can only take the values zero or one, whereas for bosons, like the scalar particle we considered previously, field operators are commuting and so any number of them can be in the same state. By calculating the total charge, it can be shown that the operator $b_s^\dagger(\mathbf{p})$ creates a **particle** with momentum $\mathbf{p}$ and spin s, while the operator $d_s^\dagger(\mathbf{p})$ creates an **antiparticle** with momentum $\mathbf{p}$ and spin s. Actually, the probability current (5.19) supplies (homework) the **charge operator**

$$\begin{aligned} Q &= -e\int d^3r :j^0(x): \\ &= -e\int d^3r :\overline{\psi}(x)\gamma^0\psi(x): \\ &= -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_p}\left[b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p})d_s(\mathbf{p})\right] \end{aligned} \tag{5.121}$$

where the terms $b_s^\dagger(\mathbf{p})b_s(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})d_s(\mathbf{p})$ are the particle number operators of momentum $\mathbf{p}$ and spin s. Thus, the application of the charge operator on the state $|\mathbf{p}', s'\rangle_+$ given by (5.115) provides

$$\begin{aligned} Q|\mathbf{p}', s'\rangle_+ &= -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} b_s^\dagger(\mathbf{p})b_s(\mathbf{p})|\mathbf{p}', s'\rangle_+ \\ &= -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} b_s^\dagger(\mathbf{p})b_s(\mathbf{p})b_{s'}^\dagger(\mathbf{p}')|0\rangle \\ &= -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} b_s^\dagger(\mathbf{p})\left[-b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p}) + (2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}\right]|0\rangle \\ &= -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} (2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}b_s^\dagger(\mathbf{p})|0\rangle \\ &= -e\, b_{s'}^\dagger(\mathbf{p}')|0\rangle = -e|\mathbf{p}', s'\rangle_+ \ . \end{aligned} \tag{5.122}$$

The same operation carried out on the state $|\mathbf{p}', s'\rangle_-$ gives the result

$$Q|\mathbf{p}', s'\rangle_- = e|\mathbf{p}', s'\rangle_- \ . \tag{5.123}$$

We deduces that $b_s^\dagger(\mathbf{p})$ creates a particle of charge $-e$ (electron) and momentum $\mathbf{p}$, whereas $d_s^\dagger(\mathbf{p})$ creates a particle of charge $+e$ (positron) and momentum $\mathbf{p}$ named antiparticle.

At the end of this study of free quantum fields, it is worthwhile to summarize the key steps of field theory.

- Fields are first defined as classical quantities given by a Lagrangian or the corresponding Euler-Lagrange equations. The values of the classical field $\Phi(\mathbf{r}, t)$ at every space point play the role of dynamical coordinates with an uncountably infinite number of degrees of freedom labeled by $\mathbf{r}$.

- On one side, a field $\Phi(\mathbf{r}, t)$ interpreted as probability amplitude and governed by the evolution equation belongs to (relativistic) quantum mechanics. In this case, the position coordinate $\mathbf{r}$ is an operator. Examples of such systems are given by the Dirac hydrogen atom (5.53) and the relativistic Coulomb scattering (5.75).

- On the other side, a field $\Phi(\mathbf{r}, t)$ interpreted as an operator acting on a Fock space and obeying equal-time commutation or anti-commutation relations, leads to quantum field theory that becomes automatically a quantum-mechanical theory of many particles. In this case, the position coordinate $\mathbf{r}$ is merely a parameter. An example of such field is supplied by the photon field (4.40) giving rise to the spontaneous emission.

We have quantized the scalar field, the electromagnetic field and the fermion field. But, for the moment, all these quantum fields have been considered as free. The real difficulties of QFT appear when the fields are interacting. This problem requires perturbation expansions and will be addressed in the next chapters.

Chapter 6

Interacting Fields

So far, we have been basically dealing with free quantum fields. The equations of motion have been given and solved in terms of Fourier transforms. As long as no interaction occurs, the dynamics of a free quantum field is rather trivial : a state is given at some initial time and the system remains in this state for all subsequent times. We now intend to describe more realistic situations in which the field changes its state because of interactions as, for instance, in the spontaneous emission.

Field theory was developed to describe scattering phenomena in which particles interact each other. In a scattering experiment, particles come in from a long time in the past, interact, and head out towards infinity. The goal of QFT is to calculate the quantum mechanical scattering amplitude for an initial state to change into a final state and to determine the transition probability or cross-section. Such calculations were already performed for the relativistice transition probability (5.88). However, in QFT, the non-commutativity of the field operators requires more technical means. The breakthrough that made field theory calculations feasible came from the Feynman's idea to write the perturbation expansion with diagrams. However, the formulas corresponding to Feynmann diagrams are diverging and need special treatment which makes sense to these formally infinite terms. This procedure is called **renormalization**.

6.1 Interaction Picture and S-matrix

In quantum mechanics, there are various ways to treat the time evolution of a state. These different manners, called pictures, are equivalent in the sense that, at the end, all observable quantities have the same matrix elements. The Schrödinger and Heisenberg pictures are probably the most familiar. Here, we will use a third one, the interaction picture. Let us consider a physical system described by the time-independent Hamiltonian

$$H = H_0 + \bar{H}_I \tag{6.1}$$

where H_0 is the free field part and $\bar{H}_I$ the interacting part. The evolution of the quantum state $|\psi_S(t)\rangle$ is described by the equation

$$i\frac{d}{dt}|\psi_S(t)\rangle = H|\psi_S(t)\rangle \tag{6.2}$$

whose solution, for a time-independant Hamiltonian, can be written

$$|\psi_S(t)\rangle = e^{-iH(t-t')}\ |\psi_S(t')\rangle. \tag{6.3}$$

The **interaction picture** or **Dirac picture** is defined by the new quantum state

$$|\psi(t)\rangle = e^{iH_0 t}|\psi_S(t)\rangle \tag{6.4}$$

that satifies (check it !) the evolution equation

$$i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle \tag{6.5}$$

containing the time-dependent interaction Hamiltonian

$$H_I(t) = e^{iH_0 t}\bar{H}_I e^{-iH_0 t}. \tag{6.6}$$

The time evolution of $|\psi(t)\rangle$ can also be described by a unitary operator $U_I(t,t')$ depending on H and supplying the ket state

$$|\psi(t)\rangle = U_I(t,t')|\psi(t')\rangle \tag{6.7}$$

where $U_I(t,t) = I$. With the above definitions, it is possible to write

$$\begin{aligned} |\psi(t)\rangle &= e^{iH_0 t}|\psi_S(t)\rangle \\ &= e^{iH_0 t}e^{-iH(t-t')}|\psi_S(t')\rangle \\ &= e^{iH_0 t}e^{-iH(t-t')}e^{-iH_0 t'}|\psi(t')\rangle \end{aligned} \tag{6.8}$$

and deduce the explicit form of the **evolution operator**

$$U_I(t,t') = e^{iH_0 t}e^{-iH(t-t')}e^{-iH_0 t'}. \tag{6.9}$$

For describing the scattering processes, we consider at $t \to \pm\infty$ free (non-interacting) particles in the initial and final eigenstates $|\phi_i\rangle$, $|\phi_f\rangle$ of the free field Hamiltonian H_0

$$H_0|\phi_\alpha\rangle = E_\alpha|\phi_\alpha\rangle \qquad \alpha = i, f \ . \tag{6.10}$$

The unitary operator which describes the evolution of states given at $t' \sim -\infty$ to states detected at $t \sim \infty$ is written S and is called **scattering matrix** or S**-matrix**. Then, the transition from the state $S|\phi_i\rangle$ into the free final state $|\phi_f\rangle$ is defined by the probability amplitude

$$S_{fi} = \langle\phi_f|S|\phi_i\rangle \ . \tag{6.11}$$

On the other side, by considering the evolution operator (6.7), we see that the transition can also be described by the following time limits

$$\lim_{t\to\infty}\lim_{t\to-\infty}\langle\psi(t)|U_I(t,t')|\psi(t')\rangle. \tag{6.12}$$

Thus, if we define initial and final states as

$$|\phi_i\rangle = |\psi(-\infty)\rangle \qquad |\phi_f\rangle = |\psi(+\infty)\rangle \tag{6.13}$$

and identify the time limits (6.12) with the definition of the probability amplitude (6.11), we arrive at a formal expression for the S**-matrix**

$$S \equiv \lim_{t\to\infty}\lim_{t'\to-\infty} U_I(t,t') \ . \tag{6.14}$$

6.2 Chronological Product and Dyson Expansion

The evaluation of the S-matrix limit (6.14) is quite complicated. As we will see, it can be replaced by a series expansion. From the definition (6.9), it is easy to verify (do it !) that the evolution operator $U_I(t,t_0)$ obeys a Schrödinger-like equation

$$i\frac{d}{dt}U_I(t,t_0) = H_I(t)U_I(t,t_0) \tag{6.15}$$

where $U_I(t_0,t_0) = I$ and $H_I(t)$ is the interaction Hamiltonian (6.6). Thanks to the interaction picture, only the interaction Hamiltonian appears in Eq. (6.15) which can be written as an equivalent integral equation

$$U_I(t,t_0) = I - i\int_{t_0}^{t} dt_1 H_I(t_1)U_I(t_1,t_0) \tag{6.16}$$

suitable for iterative calculations. Starting from the initial replacement $U_I(t_1,t_0) \to I$, we successively substitute the new solutions into (6.16) and obtain the series[1]

$$\begin{aligned} U_I(t,t_0) &= I - i\int_{t_0}^{t} dt_1 H_I(t_1)\left[I - i\int_{t_0}^{t_1} dt_2 H_I(t_2) + \cdots\right] \\ &= I - i\int_{t_0}^{t} dt_1\ H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\ H_I(t_1)H_I(t_2) + \cdots \end{aligned} \tag{6.17}$$

If we want to calculate the limits $t \to \infty$ and $t_0 \to -\infty$, we must rearrange the integrals in such way to get the same upper limits. The many possible ways to write formula (6.17) are organized with respect to the **chronological product**[2] or **time-ordered product**. For two bosons operators, it is defined as

$$T\Big[H_I(t_1)H_I(t_2)\Big] = \theta(t_1-t_2)H_I(t_1)H_I(t_2) + \theta(t_2-t_1)H_I(t_2)H_I(t_1) \tag{6.19}$$

where $\theta(x)$ is the Heaviside function. For two fermion operators, because of their anticommuting nature, the time-ordered product must allow for the change of sign when interchanging two fermion fields. It reads

$$T\Big[H_I(t_1)H_I(t_2)\Big] = \theta(t_1-t_2)H_I(t_1)H_I(t_2) - \theta(t_2-t_1)H_I(t_2)H_I(t_1). \tag{6.20}$$

After a rearrangement of the integrals in (6.17), according to the procedure sketched above, it is possible to show that it results a series called **Dyson expansion**

$$S = \lim_{t\to\infty}\lim_{t_0\to-\infty} U_I(t,t_0) = I + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} dt_1\cdots\int_{-\infty}^{+\infty} dt_n\ T\Big[H_I(t_1)\cdots H_I(t_n)\Big]. \tag{6.21}$$

[1] This series will be much more difficult to handle than those in (3.17) or (5.78) because here we are dealing with operators.

[2] General definition : given $S(n)$ the permutation group of n elements $\{1\cdots n\}$, the chronological product of the operators $H_I(t_1),\cdots,H_I(t_n)$ is defined as

$$T[H_I(t_1)\cdots H_I(t_n)] = \sum_{\pi\in S(n)} \epsilon_\pi\ \theta(t_{\pi(1)}-t_{\pi(2)})\cdots\theta(t_{\pi(n-1)}-t_{\pi(n)})H_I(t_{\pi(1)})\cdots H_I(t_{\pi(n)}) \tag{6.18}$$

where ϵ_π denotes the signature of the permutation of the fermion operators involved in this product.

The proof of the Dyson expansion is quite difficult. We can nevertheless check it for $n = 2$ by performing the following calculations

$$\begin{aligned}
&\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\left[H_I(t_1)H_I(t_2)\right] \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left[\theta(t_1 - t_2)H_I(t_1)H_I(t_2) + \theta(t_2 - t_1)H_I(t_2)H_I(t_1)\right] \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2)H_I(t_1) \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I(t_1)H_I(t_2) \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) \\
&= 2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2).
\end{aligned}$$

We have used the definition of the θ-function that introduces the following changes in the dt_2 integration

$$t_1 - t_2 > 0 \quad t_2 \in [t_0, t_1] \longrightarrow \int_{t_0}^{t_1} dt_2 \qquad\qquad t_2 - t_1 > 0 \quad t_2 \in [t_1, t] \longrightarrow \int_{t_1}^{t} dt_2 \; . \quad (6.22)$$

In the second term, we have exchanged the dummy variables t_1, t_2, and permuted the dt_2, dt_1 hierarchy of integration according to the rule of Figure 6.1 .

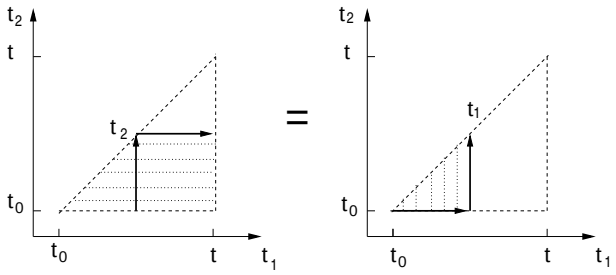

Figure 6.1: Hierarchy of dt_2, dt_1 integrations

The Dyson series is a perturbative series whose convergence cannot be demonstrated. In QED, we observe that, because of the smallness of the coupling constant[3] α, called fine structure constant, the first terms of the series already show good agreement with experimental results. One generally says that the series is asymptotically diverging. The next important task is to evaluate the chronological product T in the Dyson expansion.

[3]The strength of the field interaction is characterized by a parameter called coupling constant. This parameter is very small in QED ($\alpha \approx 1/137$), but is high for strong interaction ($g \approx 1$). The validity of the perturbative treatment is obviously depending on the size of this parameter.

6.3 Contractions and Wick's Theorem

The Dyson expansion is a lengthy expression that can be calculated term by term. However, each term contains the chronological product $T[A_1 \cdots A_n]$ whose evaluation is non-trivial. This difficulty can be partially surmounted by means of the Wick's theorem. Let us first give two definitions :

- The **normal product** of n operators

$$:A_1 \cdots A_n: \tag{6.23}$$

  means that all creation operators stand to the left of all annihilation operators. It has already been defined in (1.62). Remember that an odd number of interchanges of pairs of fermionic operators changes the sign.

- The **contraction** of two operators taken at different space-time points is defined by the vacuum-expectation value of the chronological product

$$\overbrace{A_1 A_2} = \langle 0|T[A_1 A_2]|0\rangle \ . \tag{6.24}$$

With these two definitions of normal ordering and contraction, the chronological product can be nicely expanded by means of the Wick's theorem.

Wick's theorem

$$\begin{aligned} T[A_1 \cdots A_n] \ &= \ :A_1 \cdots A_n: \\ &+ :\overbrace{A_1 A_2} A_3 A_4 \cdots A_n: + :\overbrace{A_1 A_2 A_3} A_4 \ \cdots A_n: \\ &\qquad + \text{all simple contractions} \\ &+ :\overbrace{A_1 A_2}\,\overbrace{A_3 A_4} \cdots A_n: + :\overbrace{A_1 A_2 A_3 A_4} \cdots A_n: \\ &\qquad + \text{all double contractions} \\ &+ \text{all other possible kinds of contractions.} \end{aligned} \tag{6.25}$$

The proof can be done by induction (see textbooks).

We easily verify this theorem for two scalar fields. A real scalar field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \left[a(\mathbf{p})e^{-ipx} + a^\dagger(\mathbf{p})e^{ipx}\right] \tag{6.26}$$

can be separated into two terms $\phi(x) = \phi^-(x) + \phi^+(x)$: one with negative frequency containing only annihilation operators, the other with positive frequency containing only creation operators. Thus, for a scalar field taken at any two space-time points $\phi_1 = \phi(x_1)$ and $\phi_2 = \phi(x_2)$ with positive and negative frequency components ϕ_i^- and ϕ_i^+ $i = 1, 2$, we can carry out the expansion

$$\begin{aligned} \phi_1\phi_2 \ &= \ (\phi_1^- + \phi_1^+)(\phi_2^- + \phi_2^+) \\ &= \ \phi_1^-\phi_2^- + \phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_1^-\phi_2^+ \ , \end{aligned} \tag{6.27}$$

and write the normal product expression

$$\begin{aligned} :\phi_1\phi_2: &= \phi_1^-\phi_2^- + \phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- \\ &= \phi_1\phi_2 - \left[\phi_1^-, \phi_2^+\right] \ . \end{aligned} \tag{6.28}$$

The product of the scalar fields taken at two different points then reads

$$\phi_1\phi_2 =:\phi_1\phi_2: + \left[\phi_1^-, \phi_2^+\right] \tag{6.29}$$

and the contraction of the two operators can thus be written

$$\begin{aligned} \overbrace{\phi_1\phi_2} &= \langle 0|T[\phi_1\phi_2]|0\rangle = \langle 0|\phi_1\phi_2|0\rangle\theta(t_1-t_2) + \langle 0|\phi_2\phi_1|0\rangle\theta(t_2-t_1) \\ &= \langle 0|\left[\phi_1^-, \phi_2^+\right]|0\rangle\theta(t_1-t_2) + \langle 0|\left[\phi_2^-, \phi_1^+\right]|0\rangle\theta(t_2-t_1) \\ &= \left[\phi_1^-, \phi_2^+\right]\theta(t_1-t_2) + \left[\phi_2^-, \phi_1^+\right]\theta(t_2-t_1) \ , \end{aligned} \tag{6.30}$$

where we have used the fact that the commutator[4] is a c-number (check it!) and that the vacuum is normalized $\langle 0|0\rangle = 1$. Then, with the product (6.29) and the contraction (6.30), it is straightforward to calculate the chronological product of the scalar fields ϕ_1 and ϕ_2 and arrive at the Wick's expansion

$$\begin{aligned} T[\phi_1\phi_2] &= \phi_1\phi_2\theta(t_1-t_2) + \phi_2\phi_1\theta(t_2-t_1) \\ &= \left(:\phi_1\phi_2: + \left[\phi_1^-, \phi_2^+\right]\right)\theta(t_1-t_2) + \left(:\phi_2\phi_1: + \left[\phi_2^-, \phi_1^+\right]\right)\theta(t_2-t_1) \\ &= :\phi_1\phi_2: + \overbrace{\phi_1\phi_2} \ , \end{aligned} \tag{6.31}$$

where in the last line we have used the property $:\phi_1\phi_2: = :\phi_2\phi_1:$. For fermion fields, we proceed in the same way (homework). However, because of the anticommuting nature of the fermion field, we must define the normal product with a minus sign

$$:\psi_a\psi_b: \ = \psi_a^-\psi_b^- + \psi_a^+\psi_b^+ + \psi_a^+\psi_b^- - \psi_b^+\psi_a^- = \psi_a\psi_b - \{\psi_a^-, \psi_b^+\} \qquad a, b = 1, \cdots, 4 \ . \tag{6.32}$$

Finally, it is interesting to calculate the vacuum-expectation value of the product of the scalar field $\phi(x)$ with the creation operator $a^\dagger(\mathbf{k})$. It is easy to verify (do it !), that we obtain an ingoing plane wave

$$\langle 0|\phi(x)a^\dagger(\mathbf{k})|0\rangle = e^{-ikx}. \tag{6.33}$$

The outgoing plane wave is given by

$$\langle 0|a(\mathbf{k})\phi(x)|0\rangle = e^{+ikx}. \tag{6.34}$$

Contractions are very important quantities. They are related to non-equal time commutators or anticommutators. In the next chapter we will discuss their relation with Feynman propagators we have already met earlier, for instance in (3.21).

[4]Speaking generaly, we can show that the commutator $[\phi(x_1), \phi(x_2)]$ is non-equal zero for time-like separation $(x_1 - x_2)^2 > 0$. It nevertheless vanishes at equal time, where $(x_1 - x_2)^2 = -(\mathbf{r}_1 - \mathbf{r}_1)^2 < 0$. Thus, owing to the Lorentz invariance of our fields, it is also zero for space-like separation $(x_1 - x_2)^2 < 0$ as required by a causal theory.

6.4 Propagators

The Wick theorem allows to express the chronological product in terms of normal products and contractions. Among the various possible contractions, the propagators play a central role. In the previous chapters, these quantities have already been defined as Green's functions relative to a given operator and have been subject to a particular choice of integration path. Here, they are defined by the contraction of two fields of the same kind, taken at different space-time points, and multiplied by a conventional $(-i)$ factor. The calculations below will show the equivalence of these two definitions. Moreover, the chronological product that enters the definition of the contraction will justify the choice of the causal paths depicted in Figures 3.1 and 5.1.

Boson propagator

$$\begin{aligned}\Delta_F(x-y) &= -i\,\overbrace{\phi(x)\phi}(y) = -i\langle 0|T[\phi(x)\phi(y)]|0\rangle \\ &= -i\left[\theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle\right]\end{aligned}$$

The first term gives

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\int \frac{d^3p'}{(2\pi)^3 2\epsilon_{\mathbf{p}'}}\langle 0|[a(\mathbf{p})e^{-ipx}+a^\dagger(\mathbf{p})e^{ipx}] \\ &\qquad\qquad [a(\mathbf{p}')e^{-ip'y}+a^\dagger(\mathbf{p}')e^{ip'y}]|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\int \frac{d^3p'}{(2\pi)^3 2\epsilon_{\mathbf{p}'}}\langle 0|a(\mathbf{p})a^\dagger(\mathbf{p}')|0\rangle\; e^{-ipx+ip'y} \\ &= \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\int \frac{d^3p'}{(2\pi)^3 2\epsilon_{\mathbf{p}'}}(2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')e^{-ipx+ip'y} \\ &= \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}e^{-ip(x-y)},\end{aligned}$$

and the second leads to a similar expression with a positive exponent. The sum of these two terms, each one multiplied by the step function, corresponds to the result already obtained in (3.22) for the Green's function $G_F(x-y)$. Thus, looking at the expression (3.21), the covariant form of the boson propagator can immediately be written

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon}. \tag{6.35}$$

Photon propagator

$$\begin{aligned}D_F^{\mu\nu}(x-y) &= -i\,\overbrace{A^\mu(x)A^\nu}(y) = -i\langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle \\ &= -i\left[\theta(x^0-y^0)\langle 0|A^\mu(x)A^\nu(y)|0\rangle + \theta(y^0-x^0)\langle 0|A^\nu(y)A^\mu(x)|0\rangle\right]\end{aligned}$$

As for the covariant quantization of the electromagnetic field, the calculation of the covariant photon propagator brings out some difficulties. Here, we simply give the result

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{(-g^{\mu\nu})}{k^2+i\epsilon}\; e^{-ik(x-y)}. \tag{6.36}$$

Fermion propagator

$$\begin{aligned} S^F_{ab}(x-y) &:= -i\,\overbrace{\psi_a(x)\overline{\psi}_b}(y) := -i\langle 0|T[\psi_a(x)\overline{\psi}_b(y)]|0\rangle \\ &= -i\left[\theta(x^0-y^0)\langle 0|\psi_a(x)\overline{\psi}_b(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\psi}_b(y)\psi_a(x)|0\rangle\right] \end{aligned}$$

The minus sign between the two terms comes from the definition of the chronological product for anticommuting fermion fields. Moreover, the spinor indices $a, b = 1, \cdots, 4$ must be explicitly written in order to get rid of the non-commutativity of the spinor matrices. The calculation is performed (homework) as for the boson propagator and by using the closure relations (5.41). It leads to the result (5.84) which is equivalent to the covariant fermion propagator

$$S^F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{(\not{p}+m)}{p^2-m^2+i\epsilon}\, e^{-ip(x-y)}. \tag{6.37}$$

We now have most of the technical means of QFT. We could apply them to various domains of particle physics like QED or QCD (Quantum ChromoDynamics). Because of lack of time, only one simple example of QED will be addressed in the next chapter.

Chapter 7

Feynman Diagrams in QED

One should now consider the various quantum fields, define their mutual interactions and carry out a systematic analysis of the terms of the Dyson expansion. But, the setting up of this program is a huge task that falls outside of the scope of this introductory course. It is also the case for a thorough discussion of Feynman diagrams. However, in the light of an example taken from QED, we will show how to use the Dyson series and the Wick's theorem, and also how to interpret the resulting formulas as Feynman diagrams.

7.1 Dyson Expansion Applied to QED : an Example

QED provides very accurate results. For instance, the calculation of the anomalous magnetic moment of the electron (up to 891 Feynman diagrams) combined with the experimental measurement of the g-factor yields the most precise value of the fine structure constant α, namely $1/\alpha = 137,035999710(96)$. More precisely, experimental and theoretical values of the g-factor are

$$\begin{aligned} g_{exp} &= 2(1 + 1159652209(31) \times 10^{-12}) \\ g_{theo} &= 2(1 + 1159652478(144) \times 10^{-12}). \end{aligned} \tag{7.1}$$

The first order correction supplied by one-loop diagrams already gives a good value

$$g = 2(1 + \frac{\alpha}{2\pi}).$$

Of course, a systematic approach of these calculations cannot be undertaken in this short introductory lecture. Nevertheless, the analysis of at least one example of one-loop diagram may be helpful in the understanding of the Feynman diagram technique.

The minimal coupling between fermion field ψ (electrons) and electromagnetic field A_μ (photons) can be described by the interaction Hamiltonian

$$\begin{aligned} H_I(t) &= \int d^3r \ \mathcal{H}_I \\ &= \int d^3r \ j^\mu(x) A_\mu(x) \\ &= \int d^3r \left[-e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \right]. \end{aligned} \tag{7.2}$$

Then, the first two terms of the Dyson expansion take the explicit form

$$\begin{aligned} S &= S^{(1)} + S^{(2)} + \cdots \\ &= -i \int dt_1 H_I(t_1) - \frac{1}{2} \int dt_1 \int dt_2 \; T\left[H_I(t_1) H_I(t_2)\right] + \cdots \\ &= ie \int d^4x \; (\overline{\psi} \not{A} \psi)_x - \frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; T\left[(\overline{\psi} \not{A} \psi)_{x_1} (\overline{\psi} \not{A} \psi)_{x_2}\right] + \cdots \end{aligned} \tag{7.3}$$

A complete Wick's expansion of the above expression is already a large task. Fortunately, the classification of the various terms of the Wick's theorem can be facilitated by the use of Feynman diagrams. We shall give hereinafter a mainly qualitative discussion of the role of these diagrams. No systematic analysis of connected or disconnected diagrams will be performed. With the $S^{(1)}$ term, the formula (5.90) can be recovered, but here for field operators. The expectation value of $S^{(1)}$ between ingoing and outgoing electron-states $|\mathbf{p}, s\rangle_+ = b_s^\dagger |0\rangle$ leads to the matrix element

$$ {}_+\langle \mathbf{p}, s | S^{(1)} | \mathbf{p}', s' \rangle_+ = ie \int d^4x \langle 0 | b_s(\mathbf{p}) (\overline{\psi} \not{A} \psi)_x b_{s'}^\dagger(\mathbf{p}') | 0 \rangle, \tag{7.4}$$

whose evaluation (homework) gives

$$ {}_+\langle \mathbf{p}, s | S^{(1)} | \mathbf{p}', s' \rangle_+ = -(2\pi)^4 \delta(\mathbf{p} - \mathbf{p}') \; \overline{u}_s(\mathbf{p}) (-ie\gamma^\mu) u_{s'}(\mathbf{p}') A_\mu(p - p'). \tag{7.5}$$

The next step consists of applying the Wick's theorem to the $S^{(2)}$ term. Among the various possible simple, double and triple contractions of non-equal time operators, we consider, for instance, one typical double contraction called **vacuum-polarization**

$$S_{pol}^{(2)} = \frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; :\overbrace{(\overline{\psi} \not{A} \psi)_{x_1} (\overline{\psi} \not{A} \psi)_{x_2}}: \; . \tag{7.6}$$

It can be written in terms of spinor components

$$\begin{aligned} S_{pol}^{(2)} &= \frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; \overbrace{\psi_{a'}(x_1) \overline{\psi}_b}(x_2) \overbrace{\overline{\psi}_a(x_1) \psi_{b'}}(x_2) \gamma^\mu_{aa'} \gamma^\nu_{bb'} :A_\mu(x_1) A_\mu(x_2): \\ &= (-1) \frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; \gamma^\mu_{aa'} \overbrace{\psi_{a'}(x_1) \overline{\psi}_b}(x_2) \gamma^\nu_{bb'} \overbrace{\psi_{b'}(x_2) \overline{\psi}_a}(x_1) :A_\mu(x_1) A_\mu(x_2): , \end{aligned}$$

where the minus sign comes from the odd number of anticommutations of fermion fields. The Einstein summation convention is assumed for repeated indices a, b, a', b'. The sum over spinor indices can be replaced by the trace[1] Tr and gives

$$\begin{aligned} S_{pol}^{(2)} &= -\frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; Tr\left[\gamma^\mu \overbrace{\psi(x_1) \overline{\psi}}(x_2) \gamma^\nu \overbrace{\psi(x_2) \overline{\psi}}(x_1)\right] :A_\mu(x_1) A_\nu(x_2): \\ &= -\frac{e^2}{2} \int d^4x_1 \int d^4x_2 \; Tr\left[\gamma^\mu i S^F(x_1 - x_2) \gamma^\nu i S^F(x_2 - x_1)\right] :A_\mu(x_1) A_\nu(x_2): . \end{aligned} \tag{7.7}$$

[1]The relation between the sum over spinor components and the trace can be seen by a direct application of the row-column multiplication rule for matrices :

$$\sum_a \sum_{a'bb'} \gamma^\mu_{aa'} S_{a'b} \gamma^\nu_{bb'} S_{b'a} = \sum_a [\gamma^\mu S \gamma^\nu S]_{aa} = Tr[\gamma^\mu S \gamma^\nu S].$$

For the calculation of the trace, the properties (5.99) are useful. At first sight, the above formula seems relatively complicated. However, the different factors entering its composition show a certain symmetry with respect to the space-time points. As we will see in the next section, this symmetry can be used for interpreting them as particles and interactions.

7.2 Feynman Diagrams

The integrals (7.7) written in x-configuration could be graphically represented. However, it is more usual to implement the correspondence between formula and diagram in p-space representation. With the fermion propagator (6.37), the expression (7.7) becomes

$$\begin{aligned} S^{(2)}_{pol} &= -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 \; Tr\left[\gamma^\mu i S^F(x_1-x_2)\gamma^\nu i S^F(x_2-x_1)\right] :A_\mu(x_1)A_\mu(x_2): \\ &= -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 \; Tr\left[\gamma^\mu \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not p + m)}{p^2-m^2+i\epsilon}\; e^{-ip(x_1-x_2)}\right. \\ &\qquad \left.\gamma^\nu \int \frac{d^4p'}{(2\pi)^4}\,\frac{i(\not p' + m)}{p'^2-m^2+i\epsilon}\; e^{-ip'(x_2-x_1)}\right] :A_\mu(x_1)A_\nu(x_2):. \end{aligned} \tag{7.8}$$

The fields $A_\mu(x)$ are not contracted. We contract them with ingoing and outgoing photons: $a^\dagger_{n'}(\mathbf{k}')|0\rangle$, $\langle 0|a_n(\mathbf{k})$ $n, n' = 1, 2$. For example an ingoing photon gives rise to the expression

$$\begin{aligned} \langle 0|A_\nu(x_2)a^\dagger_{n'}(\mathbf{k}')|0\rangle &= \langle 0|A_\nu(x_2)a^\dagger_{n'}(\mathbf{k}')|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \sum_{m=1}^{2} e^m_\nu(\mathbf{p}) e^{-ipx_2}\langle 0|a_m(\mathbf{p})a^\dagger_{n'}(\mathbf{k}')|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ipx_2} \sum_{m=1}^{2} e^m_\nu(\mathbf{p})(2\pi)^3 2\omega_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{k}')\delta_{mn'} \\ &= e^{n'}_\nu(\mathbf{k}')e^{-ik'x_2}, \end{aligned} \tag{7.9}$$

and also for an outgoing photon.

$$\langle 0|a_n(\mathbf{k})A_\mu(x_1)|0\rangle = e^n_\mu(\mathbf{k})e^{ikx_1}. \tag{7.10}$$

The space-time integrals in (7.8) are easily calculated. One first brings together the exponential functions appearing in (7.8) and also in the ingoing and outgoing photons

$$e^{i(-p+p'+k)x_1} \qquad e^{i(p-p'-k')x_2}.$$

Then, we see that the d^4x_1 and d^4x_2 integrations provide two $\delta^{(4)}(p-p')$ functions. One of them absorbs the d^4p' integration and leads to the expression

$$\begin{aligned} \langle S^{(2)}_{pol}\rangle = \frac{1}{2}\delta^{(4)}(k-k') \\ e^n_\mu(\mathbf{k})\int d^4p \; Tr\left[(-ie\gamma^\mu)\frac{i(\not p + m)}{p^2-m^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\not p - \not k + m)}{(p-k)^2-m^2+i\epsilon}\right] e^{n'}_\nu(\mathbf{k}'). \end{aligned} \tag{7.11}$$

The other $\delta^{(4)}(k-k')$ function represents the energy-momentum conservation. Integral and trace stand respectively for the sum over all internal momenta and spins. The nice arrangement of the different factors suggests a graphical representation.

To each factor of the formula (7.11), it is possible to assign a graphical symbol in the following way :

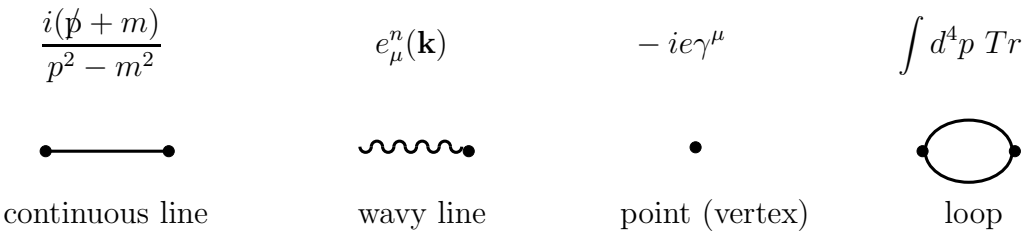

Figure 7.1: Graphical symbols

As a first simple example, one could give the graphical transcription of formula (7.5). In this case, the spinors will be represented by straight lines and the electromagnetic field by a wavy line ending by a cross as depicted in Figure 7.2 .

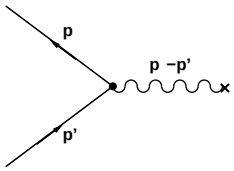

Figure 7.2: External electromagnetic field

From the **Feynman rules** given in Figure 7.1, it is easy to identify the formula (7.11) with the **vacuum-polarization diagram** depicted in Figure 7.3. Reciprocally, for a given graph, the corresponding formula can be directly written.

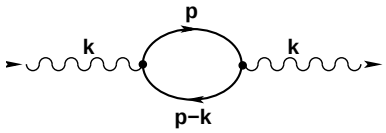

Figure 7.3: Vacuum-polarization diagram

The diagram of Figure 7.3 provides a one-loop correction to the photon propagator. However, it appears that the corresponding integral (7.11) is diverging in the following way

$$\int \frac{d^4p}{p^2} \ \sim \ \int_0^\infty p\ dp\ . \tag{7.12}$$

These kind of **ultraviolet divergences** are common to all terms of the perturbation expansion. A general procedure, called **renormalization**, has been developped in order to overcome this problem of divergences. It consists, first to reguralize the integral, for instance with a momentum cut-off Λ, then to extract a finite part, and finally to absorb the remaining infinity into a re-definition of the measured charge (charge renormalization). But all that is another story.

Chapter 8

Appendix : Exercices

1. The periodic real scalar field $\phi(x,t)$ defined on the interval $x \in [0,\ell]$ is expanded in a Fourier series

$$\phi(x,t) = \frac{1}{\sqrt{\ell}} \sum_k \phi_k(t) e^{ikx} \qquad k = \frac{2\pi n}{\ell} \ , \ n \in \mathbb{Z} \ .$$

Verify the following relations

a) $\frac{1}{\ell} \int_0^\ell dx \ e^{i(k-k')x} = \delta_{kk'}$ orthonormality

b) $\int_0^\ell \phi(x,t)^2 dx = \sum_k \phi_k(t)\phi_{-k}(t)$ Parseval's relation

c) $\frac{1}{\ell} \sum_k e^{ik(x-x')} = \delta(x-x')$ closure relation .

2. The Hamiltonian of a vibrating string of length ℓ is given by the integral

$$H = \frac{1}{2} \int_0^\ell dx \left[\pi^2 + v^2 \left(\partial_x \phi\right)^2\right]$$

with $\pi = \dot{\phi}$. By using the Fourier expansion of the field

$$\phi(x,t) = \frac{1}{\sqrt{\ell}} \sum_k \left[A_k \ e^{-i(\omega_{\mathbf{k}} t - kx)} + A_k^* \ e^{+i(\omega_{\mathbf{k}} t - kx)}\right] \ ,$$

where $\omega_k = v|k|$, deduce the Hamiltonian expression

$$H = \sum_k \omega_k^2 (A_k^* A_k + A_k A_k^*) \ .$$

3. Show that the scalar quantum field $\phi(x,t)$ and its conjugate field $\pi(x,t)$ given in section 1.3 obey the equal-time commutation relations

$$[\phi(x,t), \pi(x',t)] = i\hbar\delta(x-x')$$

$$[\phi(x,t) \ , \phi(x',t)] = 0 = [\pi(x,t) \ , \pi(x',t)] \ .$$

4. Review the canonical quantization scheme of the single harmonic oscillator and list the main ideas.

5. Show that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \Phi)^2 - m^2 \Phi]$$

yields the Klein-Gordon equation

$$[\Box - m^2]\Phi = 0 \ .$$

6. From the definition of the differential, convince yourself that the functional derivative of the functional

$$F[f] = \int \mathcal{F}(f(x))\ dx$$

is given by the partial derivative of the density function $\mathcal{F}$

$$\frac{\delta F}{\delta f} = \frac{\partial \mathcal{F}}{\partial f} \ .$$

7. Consider the scalar field $\phi'(x') = \phi(x)$ and the infinitesimal Lorentz matrix

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu + \mathcal{O}(\epsilon^2) \qquad\qquad \Lambda \in \{\Lambda / \Lambda^T g \Lambda = g\} \ .$$

a) Show that $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$.

b) Verify that the Klein-Gordon action

$$S = \int d^4x \left[(\partial_\mu \phi)^2 - m^2 \phi^2\right]$$

is invariant under the infinitesimal Lorentz transformation

$$x'^\mu = (\delta^\mu_\nu + \epsilon^\mu{}_\nu) x^\nu \ .$$

c) For any Lorentz-invariant action $S = \int d^4x\ \mathcal{L}(\phi, \partial_\mu \phi)$, deduce the conserved current

$$\Theta^\mu = \frac{\epsilon_{\alpha\nu}}{2}\left[x^\alpha T^{\mu\nu} - x^\nu T^{\mu\alpha}\right]$$

where

$$T^{\mu\nu} = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu_\alpha \mathcal{L}\right] g^{\alpha\nu} \ .$$

d) Identify the conserved charge $Q^{jk} \quad j,k = 1,2,3.$

8. Show that the action

$$S = \int_\Omega d^4x \ \mathcal{L}(\phi, \partial_\mu \phi)$$

is invariant under the local phase transformation $\phi'(x) = e^{i\alpha(x)}\phi(x)$, where we assume the boundary condition $\alpha(x)\phi(x)|_{\partial\Omega} = 0$.

9. From the definition of the Hamiltonian

$$H[\phi, \pi] = \int_V d^3r \left[\pi\dot{\phi} - \mathcal{L}(\phi, \partial_\mu\phi, t)\right]$$

and by considering the variation (differential)

$$\delta H = \int_V d^3r \left[\frac{\delta H}{\delta\phi} h_\phi + \frac{\delta H}{\delta\pi} h_\pi\right] \ , \tag{8.1}$$

where the condition $h_\phi(\mathbf{r}, t)|_{\partial V} = 0$ is assumed, deduce the canonical equations

$$\dot{\phi} = \frac{\delta H}{\delta\pi} \qquad\qquad \dot{\pi} = -\frac{\delta H}{\delta\phi} \ .$$

10. Schow that the equal-time canonical commutation relations

$$[\phi(\mathbf{r}, t), \pi(\mathbf{r}', t)] = i\delta(\mathbf{r} - \mathbf{r}')$$

$$[\phi(\mathbf{r}, t), \phi(\mathbf{r}', t)] = 0 = [\pi(\mathbf{r}, t), \pi(\mathbf{r}', t)]$$

induce the following relations for the creation and annihilation operators

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{p}')\right] = (2\pi)^3 2\epsilon_{\mathbf{p}}\delta(\mathbf{p} - \mathbf{p}')$$

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 = \left[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')\right] \ .$$

11. A three-boson state is given by the ket $|\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)a^\dagger(\mathbf{p}_3)|0\rangle$.

a) Verify the orthogonality relation

$$\langle \mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3'|\mathbf{p}, \mathbf{p}_2, \mathbf{p}_3\rangle = \sum_{i=1}^{3} (2\pi)^3 2\epsilon_{\mathbf{p}_i'}\delta(\mathbf{p} - \mathbf{p}_i')\langle \mathbf{p}_1', \hat{\mathbf{p}}_i', \mathbf{p}_3'|\mathbf{p}_2, \mathbf{p}_3\rangle \ ,$$

where the momentum $\hat{\mathbf{p}}_i'$ must be ignored.

b) Deduce the action of the annihilation operator on this three-boson state

$$a(\mathbf{p})|\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle = \sum_{i=1}^{3} (2\pi)^3 2\epsilon_{\mathbf{p}_i}\delta(\mathbf{p} - \mathbf{p}_i)|\mathbf{p}_1, \hat{\mathbf{p}}_i, \mathbf{p}_3\rangle \ .$$

12. The Hamiltonian of the real free scalar field is given by

$$H = \frac{1}{2}\int d^3r \ \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \ .$$

a) Express it in terms of creation and annihilation operators

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})\right] \ .$$

b) Verify that the state $|\mathbf{p}_1, \cdots, \mathbf{p}_i, \cdots, \mathbf{p}_n\rangle$ is an eigenvector of the normal-ordered Hamiltonian $:H:$, with eigenvalue $E = \sum_{i=1}^n \epsilon_{\mathbf{p}_i}$.

13. The Lagrangian density of the free electromagnetic field is given by the expression

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the tensor field.

a) Verify that

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \qquad\qquad H = \frac{1}{2}\int d^3r(\mathbf{E}^2 + \mathbf{B}^2) \ .$$

b) Derive the Maxwell's equations for a free field

$$\partial_\mu F^{\mu\nu} = 0 \ .$$

14. Consider the electromagnetic field in free space ($j^\mu = 0$). Show that, for a given A_μ, there exists a gauge transformation $A'_\mu = A_\mu + \partial_\mu\chi$ such that the Coulomb gauge conditions $\nabla\cdot\mathbf{A}' = 0$, $A'_0 = 0$ are satisfied.

15. The quantization of the electromagnetic field in Coulomb gauge imposes a new commutation relation

$$\left[\pi_j(\mathbf{r},t), A_{j'}(\mathbf{r}',t)\right] = i\delta^{tr}_{jj'}(\mathbf{r}-\mathbf{r}') \ ,$$

where

$$\pi_j = -\dot{A}_j \quad \text{and} \quad \delta^{tr}_{jj'}(\mathbf{r}-\mathbf{r}') = \int \frac{d^3p}{(2\pi)^3}\ e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\left[\delta_{jj'} - \frac{p_j p_{j'}}{\mathbf{p}^2}\right] \ .$$

Show that this relation induces the commutator

$$\left[a_n(\mathbf{k}), a^\dagger_m(\mathbf{p})\right] = (2\pi)^3 2\omega_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{p})\delta_{nm} \qquad \omega_{\mathbf{k}} = |\mathbf{k}| \ .$$

Hint : Express the operators $a_n(\mathbf{k})$ and $a^\dagger_m(\mathbf{k})$ in terms of $A_j(x)$ and $\pi_j(x)$.

16. By using the Coulomb gauge conditions $\nabla \cdot \mathbf{A} = 0, \Phi = 0$, show that the free photon Hamiltonian takes the form

$$:H: = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \, \omega_{\mathbf{k}} \sum_{n=1}^{2} a_n^\dagger(\mathbf{k}) a_n(\mathbf{k}) \qquad\qquad \omega_{\mathbf{k}} = |\mathbf{k}| \ .$$

17. Verify the commutation relation

$$[\mathbf{r}, H_{at}] = \frac{i}{m} \mathbf{p}$$

where $\mathbf{r}$ is the position operator, $\mathbf{p}$ the momentum operator and $H_{at} = \mathbf{p}^2/2m + V(\mathbf{r})$ the atomic Hamiltonian.

18. The general form of the Dirac equation

$$\Big[\partial_t + \boldsymbol{\alpha} \cdot \nabla + iK\beta\Big]\psi(x) = 0 \qquad K \in \mathbb{R}$$

must satisfy the energy-momentum relation $\epsilon_p^2 = \mathbf{p}^2 + m^2$. From this requirement, deduce the value of K and the properties of the matrices β and $\boldsymbol{\alpha}$

$$\begin{aligned} &\beta^2 = I \\ &\alpha_j \beta + \beta \alpha_j = 0 \\ &\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \ . \end{aligned}$$

19. Derive the Dirac equation from the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\left[\overline{\psi}(x) \partial\!\!\!/ \psi(x) - \overline{\psi}(x) \overleftarrow{\partial\!\!\!/} \, \psi(x)\right] - m\overline{\psi}(x)\psi(x) \tag{8.2}$$

by considering the field components $\overline{\psi}_\alpha(x)$ and $\psi_\alpha(x)$ $\alpha = 1, \cdots, 4$ as independent variables.

20. The free electron probability current is given by the expression

$$\mathbf{j} = \overline{\psi} \boldsymbol{\gamma} \psi$$

where

$$\psi = \frac{1}{(2\pi)^{3/2}} \frac{u_s(\mathbf{p})}{\sqrt{2\epsilon_p}} \, e^{-ipx}.$$

Verify that

$$\mathbf{j} = \frac{1}{(2\pi)^3} \frac{\mathbf{p}}{\epsilon_p}.$$

21. Verify the following properties

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$$

$$\sigma_1\sigma_2 = i\sigma_3 \qquad \sigma_2\sigma_3 = i\sigma_1 \qquad \sigma_3\sigma_1 = i\sigma_2 \ .$$

of the Pauli matrices $\sigma_j \ , \ j = 1, 2, 3$.

22. Consider the Dirac spinors

$$u_s(\mathbf{p}) = \sqrt{\epsilon_p + m} \begin{bmatrix} \varphi_s \\ \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\epsilon_p + m}\ \varphi_s \end{bmatrix} \qquad v_s(\mathbf{p}) = \sqrt{\epsilon_p + m} \begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\epsilon_p + m}\ \chi_s \\ \\ \chi_s \end{bmatrix} \qquad s = 1, 2.$$

a) Verify that they satisfy the following orthonormalization conditions

$$u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = 2\epsilon_p\delta_{ss'} = v_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p})$$

$$\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = 2m\delta_{ss'} \qquad \bar{v}_s(\mathbf{p})v_{s'}(\mathbf{p}) = -2m\delta_{ss'}$$

$$u_s^\dagger(\mathbf{p})v_{s'}(-\mathbf{p}) = 0 \ .$$

b) Show that the closure relations are given by

$$\sum_{s=1}^{2} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = (\not{p} + m) \qquad \sum_{s=1}^{2} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = (\not{p} - m).$$

23. a) Show that the Dirac spin operator $\boldsymbol{\Sigma}$ has the properties

$$\alpha_1\Sigma_2 = i\alpha_3 \qquad \alpha_2\Sigma_3 = i\alpha_1 \qquad \alpha_3\Sigma_1 = i\alpha_2$$

where $\boldsymbol{\alpha} = \gamma^5\boldsymbol{\Sigma}$.

b) From the results given in the course and by using a) show that

$$\boldsymbol{\alpha}\cdot\frac{\hbar}{i}\nabla = \boldsymbol{\alpha}\cdot\mathbf{e}_r\left[\frac{\hbar}{i}\frac{\partial}{\partial r} + \frac{2i}{\hbar}\frac{(\mathbf{L}\cdot\mathbf{S})}{r}\right] \ .$$

24. For the relativistic hydrogen atom, verify that the set

$$\left\{H \ , \quad \mathbf{J}^2 \ , \quad J_z \ , \quad \mathbf{S}^2\right\}$$

forms a complete set of commuting observables.

25. a) Verify the commutator $\Big[\mathbf{J}_k\ ,\ \boldsymbol{\sigma}\cdot\mathbf{r}\Big]=0 \qquad k=1,2,3.$

b) By using a), show that

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\ \mathcal{Y}_{jm}(j\mp\tfrac{1}{2}|\vartheta,\varphi)=-\mathcal{Y}_{jm}(j\pm\tfrac{1}{2}|\vartheta,\varphi)\ .$$

26. Following the calculations of section 5.2 b) of the lecture notes, show that the integral equation for the probability amplitude is given by

$$\langle\psi_{u_{s'}}|\psi\rangle=\langle\psi_{u_{s'}}|\psi_0\rangle+ie\int d^4x'\ \overline{\psi}_{u_{s'}}(x')\,\not{A}(x')\ \psi(x').$$

27. Perform the dp_0 integration of the Feynman propagator given by

$$S_F(x-x')=\int\frac{d^4p}{(2\pi)^4}\ \frac{(\not{p}+m))}{(p^2-m^2+i\epsilon)}\ e^{-ip(x-x')}\ .$$

28. Starting from the Coulomb scattering amplitude given in the course

$$S_{fi}^{(1)}=-i\frac{Z\alpha}{(2\pi)^2\sqrt{4\epsilon_{\mathbf{p}_f}\epsilon_{\mathbf{p}_f}}}\ 4\pi\delta(\epsilon_{\mathbf{p}_f}-\epsilon_{\mathbf{p}_i})\ \bar{u}_{s_f}(\mathbf{p}_f)\frac{\gamma^0}{|\mathbf{q}|^2}\ u_{s_i}(\mathbf{p}_i)$$

with $\mathbf{q}=\mathbf{p}_f-\mathbf{p}_i$, calculate the Mott cross-section

$$\frac{d\sigma}{d\Omega}=\left(\frac{Z\alpha}{2\epsilon_{\mathbf{p}}}\right)^2\frac{\Big[1-\beta^2\sin^2(\vartheta/2)\Big]}{\beta^4\sin^4(\vartheta/2)}\ ,$$

where $\beta=v/c$, $\epsilon_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$ and ϑ is the angle between $\mathbf{p}_i$ and $\mathbf{p}_f$.

Remark that, at the limit $\beta\ll 1$, we obtain the Rutherford cross-section.

29. Express the fermionic Hamiltonian

$$H=\frac{i}{2}\int d^3r\ [\psi^\dagger\ \partial_0\psi-\partial_0\psi^\dagger\ \psi]$$

in terms of creation and annihilation operators

$$H=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left[b_s^\dagger(\mathbf{p})b_s(\mathbf{p})-d_s(\mathbf{p})d_s^\dagger(\mathbf{p})\right].$$

30. Show that the the anticommution relation

$$\{\psi_a(\mathbf{r},t),\pi_b(\mathbf{r}',t)\} = i\delta_{ab}\ \delta(\mathbf{r}-\mathbf{r}')$$

for fermion fields implies the relations

$$\left\{b_s(\mathbf{p}),b^\dagger_{s'}(\mathbf{p}')\right\} = (2\pi)^3 2\epsilon_p \delta_{ss'}\delta(\mathbf{p}-\mathbf{p}') \qquad \left\{d_s(\mathbf{p}),d^\dagger_{s'}(\mathbf{p}')\right\} = (2\pi)^3 2\epsilon_p \delta_{ss'}\delta(\mathbf{p}-\mathbf{p}')\ .$$

31. Express the charge operator

$$Q = -e\int d^3r\ :j^0(x):$$

in terms of creation and anihilation operators

$$Q = -e\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2\epsilon_p}\left[b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - d^\dagger_s(\mathbf{p})d_s(\mathbf{p})\right]\ .$$

32. Consider the time-independent Hamiltonian $H = H_0 + \bar{H}_I$ where H_0 is the free part and $\bar{H}_I$ the interacting part. For a Schrödinger state vector $|\psi_S(t)\rangle$, show that, in the interaction picture, the ket state

$$|\psi(t)\rangle = e^{iH_0t}|\psi_S(t)\rangle$$

obeys the evolution equation

$$i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$$

where $H_I(t) = e^{iH_0t}\bar{H}_I e^{-iH_0t}$.

33. For two fermion fields $\psi_1\ ,\ \psi_2$, verify the Wick's theorem

$$T[\psi_1\psi_2] = :\psi_1\psi_2: + \quad \overbrace{\psi_1\psi_2}\ .$$

34. a) Given the matrix $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, verify the following properties

$$\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu \qquad\qquad (\gamma^5)^2 = I.$$

b) Given the four-vectors $a^\mu, b^\mu, c^\mu, d^\mu$, derive the following trace properties :

1) $Tr[$ odd number of $\gamma^\mu] = 0$
2) $Tr[\not{a}\not{b}] = 4a^\mu b_\mu = 4ab$
3) $Tr[\not{a}\not{b}\not{c}\not{d}] = 4[(ab)(cd) + (ad)(bc) - (ac)(bd)]$.

35. We consider the gauge field equation $\Box A^\mu(x) = -j^\mu(x)$, where the proton current is given by $j^\mu = e\bar{\psi}_f^p \gamma^\mu \psi_i^p$. Show that the first-order approximation of the electron-proton interaction is given by the amplitude

$$\begin{aligned} S_{fi}^{(1)} &= -\frac{e^2}{(2\pi)^2}\,\delta(P_f - P_i + p_f - p_i)\,\frac{1}{\sqrt{16\;\epsilon_{\mathbf{p}_f}\epsilon_{\mathbf{p}_i}\epsilon_{\mathbf{P}_f}\epsilon_{\mathbf{P}_i}}} \\ &\quad \times \overline{u}_f(\mathbf{p}_f)\gamma^\mu u_i(\mathbf{p}_i)\frac{-g_{\mu\nu}}{(P_f - P_i)^2 + i\varepsilon}\overline{u}_f(\mathbf{P}_f)\gamma^\nu u_i(\mathbf{P}_i) \end{aligned}$$

where p and P are respectively the electron and proton four-momenta.

36. For a scalar field $\phi(x)$, calculate the following two contractions

$$\overbrace{\phi(x)a^\dagger(\mathbf{k})} \qquad\qquad \overbrace{a(\mathbf{k})\phi(x)}\ .$$

37. Calculate the free fermion propagator

$$S_{ab}^F(x-y) = -i\langle 0|T[\psi_a(x)\overline{\psi}_b(y)]|0\rangle$$

and arrive at the expression

$$S_{ab}^F(x-y) = \int \frac{d^4p}{(2\pi)^4}\;\frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}\;e^{-ip(x-y)}\ .$$