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Localization of Matter Waves in Two-Dimensional Disordered Optical Potentials

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We consider ultracold atoms in 2D disordered optical potentials and calculate microscopic quantities characterizing matter wave quantum transport in the noninteracting regime. We derive the diffusion constant as a function of all relevant microscopic parameters and show that coherent multiple scattering induces significant weak localization effects. In particular, we find that even the strong localization regime is accessible with current experimental techniques and calculate the corresponding localization length.

Introduction.—Ultracold atoms in optical potentials can be used to realize condensed matter model systems in a very versatile manner [1]. Having at hand the possibility to shape external potentials almost at will, a natural direction of investigation is the disorder driven superfluid-insulator transition [2] or the Anderson transition [3]. Interest in this transition has been renewed since the experimental observation that even a small disorder in confining fields leads to a fractioning of quasi-1D condensates in waveguide structures on atom chips [4]. Very recently, several studies of Bose condensates in speckle potentials [5] have shown an efficient suppression of 1D transport by disorder [6–8].

The physics of interacting particles in a disordered environment has already been discussed for a number of years for electrons and superfluid helium [2,9]. For cold atoms, recent contributions include a numerical study of the Bose-Hubbard and Anderson hopping models [10], a renormalization group approach in the case of off-diagonal disorder [11], a Bose-Fermi mapping for hardcore bosons [12], and a transfer matrix treatment of atomic matter waves interacting with impurity atoms in an optical lattice [13]. All these approaches study 1D systems. Bose condensates in 2D optical quasicrystal lattices were considered in Ref. [14]. In this context, our aim is to concentrate on the effect of disorder on quantum transport in the noninteracting regime, leaving the intriguing impact of interactions for later studies. Interaction effects are small in the low-density wings of expanding Bose condensates [6] that are obtained from interaction-dominated condensates in the Thomas-Fermi regime by opening the trap potential. If one insists on working in the high-density regime, single-particle dynamics can be studied by reducing the scattering length via Feshbach resonances [15], eventually reaching the ideal Bose gas regime [16]. Alternatively, one can work with spin-polarized fermions [17] whose collisions are blocked by the Pauli principle.

In this Letter, we report analytical results for the 2D dynamics of cold atoms in a far-detuned optical speckle potential. Using a perturbative Green's function approach, we calculate transport quantities relevant for the diffusive regime and derive the weak localization correction to the classical diffusion constant. The 2D geometry is particularly advantageous for measuring quantum corrections to classical transport because diffusive trajectories always return to their starting point, thus favoring localization effects. Making use of the microscopic characteristics of the speckle potential instead of effective models of disorder, we show that a highly disordered environment with strong scattering can be tailored with speckle potentials. Furthermore, we find that the strong localization threshold can be reached with current experimental techniques.

Intensity transport.—A cloud of noninteracting cold atoms is described by the single-particle Hamiltonian $H = p^2/2m + V(\mathbf{r})$ where $V(\mathbf{r})$ is a static 2D random potential after the harmonic confinement in the transport directions has been switched off as realized in Refs. [6–8]. The initial state density matrix ϱ_0 of the atomic cloud evolves in time as $\varrho(t) = U(t)\varrho_0 U^\dagger(t)$ with the evolution operator $U(t) = \exp(-iHt/\hbar)$. Meaningful matter wave transport observables involve a statistical average over all possible realizations of disorder. One important quantity is the average probability density $p(\mathbf{r}, t) = \overline{\langle\mathbf{r}|\varrho(t)|\mathbf{r}\rangle}$ of particles at point $\mathbf{r}$ and time t (a bar denotes the disorder average). Its Fourier transform $p(\mathbf{q}, \Omega) = \int d^2r \int dt\, p(\mathbf{r}, t) \exp(i\Omega t - i\mathbf{q}\mathbf{r})$ is given by

$$p(\mathbf{q}, \Omega) = \int \frac{d^2k}{(2\pi)^2} \varrho_0(\mathbf{k}, \mathbf{q})\Phi(\mathbf{k}, \mathbf{q}, \Omega), \qquad (1)$$

where $\varrho_0(\mathbf{k}, \mathbf{q}) = \langle\mathbf{k} + \mathbf{q}/2|\varrho_0|\mathbf{k} - \mathbf{q}/2\rangle$ contains all information about the initial atomic density distribution, and Φ is the intensity relaxation kernel for plane waves with on-shell energy $E = \hbar^2 k^2/2m$. In the long time and large distance limits $\Omega, q \to 0$, the 2D relaxation kernel for isotropic intensity distributions has the characteristic pole

$$\Phi(\mathbf{k}, \mathbf{q}, \Omega) = \frac{1}{-i\Omega + D(k)q^2} \qquad (2)$$

that describes a diffusion process with diffusion constant $D(k)$. Since diffusion solely relies on the local conservation

of particles and on linear response it is a very robust phenomenon. In the remainder of the Letter, we will essentially calculate the plane wave diffusion constant $D(k)$, including interference corrections, as a function of all relevant microscopic parameters. Once the diffusion constant is determined, the dynamics of any initial density distribution is obtained by integrating (1).

Speckle characteristics.—Of particular importance are the characteristics of the optical potential $V(\mathbf{r}) = \bar{V}[1 + \delta V(\mathbf{r})]$ with average $\bar{V}$ and normalized random component $\delta V(\mathbf{r})$. The only correlation function that we will need in the following is the two-point correlator $\mathcal{P}(\mathbf{r}) = \overline{\delta V(\mathbf{r}')\delta V(\mathbf{r}' + \mathbf{r})}$. We consider two-level atoms (transition frequency ω_0, transition width Γ, saturation intensity I_s) exposed to a far-detuned monochromatic speckle field $\mathcal{E}(\mathbf{r})$ at frequency $\omega_{\mathrm{L}} = ck_{\mathrm{L}}$ and detuning $\delta = \omega_{\mathrm{L}} - \omega_0$, generated from a laser source with power P. The speckle pattern is created over a surface of linear size L, the local field intensity being $I(\mathbf{r}) = \epsilon_0 c|\mathcal{E}(\mathbf{r})|^2/2$ with average value $I_{\mathrm{L}} = P/L^2$. The speckle optical dipolar potential is $V(\mathbf{r}) = \bar{V}I(\mathbf{r})/I_{\mathrm{L}}$, with $\bar{V} = (\hbar\Gamma^2/8\delta)I_{\mathrm{L}}/I_s$. This random potential derives from the Gaussian random field $\mathcal{E}(\mathbf{r})$, but is not a Gaussian variable by itself. Its pair correlator can be expressed as $\mathcal{P}(\mathbf{r}) = |\gamma(\mathbf{r})|^2$ with $\gamma(\mathbf{r}) = \epsilon_0 c\overline{\mathcal{E}(\mathbf{r}')\mathcal{E}(\mathbf{r}' + \mathbf{r})}/2I_{\mathrm{L}}$ being the normalized two-point field correlation function [18]. Its 2D Fourier transform $\mathcal{P}(\mathbf{k})$ is the speckle power spectrum.

For a 2D speckle pattern produced by monochromatic illumination of a holographic phase mask (transmission geometry) or of a rough surface (reflection geometry), the far-field correlation reads $\gamma(\mathbf{r}) = 2J_1(u)/u$ where $u = r/\zeta$ and J_1 is the first order Bessel function [19]. Here $\zeta = 1/\alpha k_{\mathrm{L}}$ is the correlation length of the speckle potential, and $\alpha = R/z \ll 1$ is the speckle aperture angle at a distance z from the speckle source with radius R. The correlation length ζ defines the intrinsic physical length scale of our system. In turn, it also defines an energy scale for the atomic dynamics, $E_\zeta = \hbar^2/m\zeta^2 = 2\alpha^2 E_R$ in terms of the more familiar recoil energy $E_R = \hbar^2 k_{\mathrm{L}}^2/2m$. The ratio $\eta = \bar{V}/E_\zeta$ measures the strength of the potential fluctuations relative to the correlation energy E_ζ. The 2D speckle power spectrum is obtained as the convolution of two identical disks, $\mathcal{P}(\mathbf{k}) = 8\mathcal{F}(k\zeta/2)$ with $\mathcal{F}(x) = [\arccos x - x\sqrt{1 - x^2}]\Theta(1 - x)$. The Heaviside distribution Θ reflects the fact that the potential is smooth on length scales smaller than ζ, but uncorrelated on distances larger than ζ.

Weak scattering regime.—Microscopic transport parameters can be calculated using standard diagrammatic Green's function techniques [20–22]. A well controlled perturbative expansion is obtained if the atomic energy E lies above the mobility edge E_c:

$$\Delta = \frac{E_c}{E} < 1; \qquad E_c = \frac{\bar{V}^2}{E_\zeta} = \eta^2 E_\zeta. \qquad (3)$$

In physical terms, this weak scattering condition can be understood as a condition for small quantum reflection from a potential bump of linear size ζ and height $\bar{V}$. It can be realized in two ways: either the atomic kinetic energy E is larger than the correlation energy E_ζ; then the potential fluctuations must be small with respect to the atomic energy, $\bar{V} < E$. This case corresponds to the classical picture of atoms flying well above small potential bumps. Or the atomic energy is smaller than the correlation energy (requiring cooling well below recoil, especially if $\alpha \ll 1$); then the fluctuations must still be smaller than E_ζ, but can be larger than the atomic energy E. This case corresponds to a quantum regime where the atom is able to tunnel through high potential bumps of linear extension ζ thanks to its large de Broglie wavelength $\lambda_{\mathrm{dB}} = 2\pi/k \gg \zeta$.

Scattering mean free path.—In the weak scattering limit, the average distance traveled by the atom between two scattering events defines the elastic scattering mean free path ℓ_s. It is related to the speckle power spectrum through

$$\frac{1}{k\ell_s} = \left(\frac{\eta}{k\zeta}\right)^2 \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{P}(k\zeta, \theta), \qquad (4)$$

where $\mathcal{P}(k\zeta, \theta) = 8\mathcal{F}(k\zeta\sin(\theta/2))$ represents the differential scattering cross-section. A plot of $k\ell_s$ as a function of the reduced atomic wave number $k\zeta$ is shown in Fig. 1 for different values of the disorder strength η. The weak scattering condition $\Delta \leq 1$ implies the bound $k\ell_s \geq 1/2\pi$ such that the shortest achievable scattering mean free path is of the order of the 2D speckle correlation length ζ itself. Figure 1 shows that the speckle potential, even though correlated on the scale ζ, can become a highly disordered scattering medium with $k\ell_s$ of order unity for sufficiently cold matter waves ($k\zeta \to 0$).
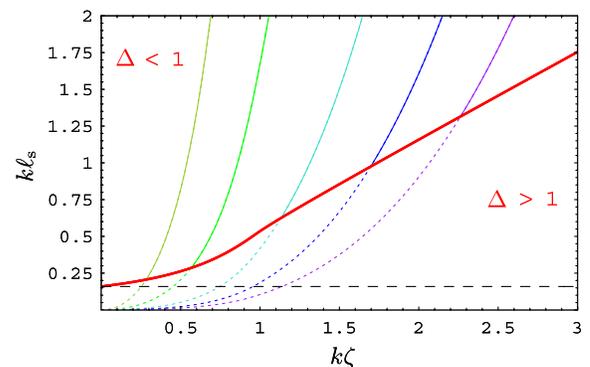

FIG. 1 (color online). Plot of $k\ell_s$ from Eq. (4) as a function of the reduced matter wave number $k\zeta$ for different values of the disorder strength $\eta = \bar{V}/E_\zeta \in \{0.2, 0.4, 0.8, 1.2, 1.6\}$ (thin curves from left to right). The thick line (red online), connecting points of $k\ell_s$ where $\Delta = 1$, i.e., $k\zeta = \sqrt{2}\eta$, indicates the limit of validity of the weak scattering condition (3). The dashed horizontal line corresponds to $k\ell_s = 1/2\pi$. A strongly disordered medium with $k\ell_s \approx 1$ is obtained for cold enough atoms.

Boltzmann diffusion constant.—While the atom propagates through the speckle field, it is scattered by potential fluctuations. After a large number of scattering events, this random walk conserving energy and particle number results in diffusive transport for matter waves with wave number k. Following the approach pioneered by Vollhardt and Wölfle [23], we set up a quantum kinetic equation for the intensity kernel defined in (1) that allows one to calculate the transport mean free path from the microscopic properties of the system. In a first step, we determine the 2D Boltzmann diffusion constant

$$D_B(k) = \frac{\hbar k \ell_{tr}(k)}{2m},\tag{5}$$

where the elastic transport mean free path ℓ_{tr} is the average distance traveled by the atom before losing memory of its initial direction. This approximation assumes that the disorder average washes out all interference effects between partial scattered waves. Scattering and transport mean free paths are then linked by the relation

$$\frac{\ell_s}{\ell_{tr}} = 1 - \frac{\int_0^{2\pi} d\theta \cos\theta \, \mathcal{P}(k\zeta, \theta)}{\int_0^{2\pi} d\theta \, \mathcal{P}(k\zeta, \theta)}.\tag{6}$$

Because of the potential correlation at small scales (Heaviside function in $\mathcal{F}(x)$), the scattering angle is bounded by $|\sin(\theta/2)| \leq 1/k\zeta$. Hence, there is no angular restriction for slow atoms ($k\zeta \leq 1$). In the ultracold regime ($k\zeta \ll 1$) isotropic scattering ($\ell_{tr} \approx \ell_s$) prevails, and the speckle potential becomes an effective δ-correlated potential. For fast atoms $k\zeta \gg 1$, the maximum scattering angle is $\theta_{max} \simeq 2/k\zeta$ and scattering is strongly peaked in the forward direction. In this case, $\ell_{tr} \approx (k\zeta)^2 \ell_s \gg \ell_s$ (strongly anisotropic scattering). For example, for Rubidium atoms with $k\zeta = 1$, $L = 2$ cm, $\alpha = 0.1$, $P = 0.25$ W, and $\delta = 10^6\Gamma$, we predict a diffusive matter wave transport with elastic scattering mean free path $\ell_s \approx 2 \ \mu$m and elastic transport mean free path $\ell_{tr} \approx 7 \ \mu$m.

Weak localization.—In phase coherent samples, the constructive interference between counter-propagating amplitudes enhances the return probability of the atomic matter wave to a given point. This weak localization correction [22,23] reduces the diffusion constant to $D = D_B - \delta D$ with

$$\frac{\delta D}{D_B} = \frac{2}{\pi} \frac{\ln(L_0/\ell_s)}{k\ell_{tr}}\tag{7}$$

in 2D. Here, the length $L_0 = \min(L, L_\phi)$ is the relevant cutoff for fully coherent multiple scattering. It is either the system size L itself, or the phase coherence length $L_\phi = \sqrt{D_B \tau_i}$ which accounts for possible phase breaking mechanisms affecting interference at a rate $\Gamma_i = \tau_i^{-1}$. For cold atoms in optical speckle potentials, one phase breaking mechanism is inelastic scattering of photons for which $\Gamma_i = \Gamma\bar{V}/\hbar\delta$. The corresponding phase coherence length L_ϕ scales as $\delta^2 I_L^{-3/2}$ while ℓ_s and ℓ_{tr} scale as $\delta^2 I_L^{-2}$. Thus,

by keeping the ratio I_L/δ fixed, all multiple scattering parameters $\bar{V}$, ℓ_s, ℓ_{tr}, and D_B are kept constant. Under this condition, changing I_L (and δ) only modifies L_ϕ and thus the interference corrections. This opens the way to use inelastic scattering to monitor the weak localization corrections in a controlled manner, just like with an external magnetic field in 2D electronic experiments on negative magnetoresistance [24].

In Fig. 2, we plot $\delta D/D_B$ as a function of the total laser power P for different values of the atomic matter wave number k at fixed laser detuning $\delta = 10^6\Gamma$. The thick black line indicates the corresponding limit of validity of the weak scattering condition $\Delta \leq 1$. The colder the atoms, the larger the quantum corrections (within the lower bound ζ/L for $k\zeta$ and for the applied laser power imposed by the diffusion condition $L_0 \geq \ell_{tr}$). Even moderate laser power assures *coherent multiple scattering* in the speckle plane and induces sizeable weak localization corrections. For $k\zeta = 2$, the relative correction $\delta D/D_B$ attains approximately 30% at $P = 0.9$ W. For $k\zeta = 1.25$, the border $\delta D = D_B$ is even reached within the region $\Delta \leq 1$ at $P = 0.5$ W: this is the strong localization onset.

Towards strong localization.—The perturbative weak localization correction (7) diverges with the cutoff length L_0, which is compatible with the scaling prediction that waves are always localized in 2D on sufficiently large coherent length scales [25]. Equation (7) is actually the result of a self-consistent theory [23] that takes into account the relevant singular terms driving the system towards the strong localization onset $\delta D/D_B \rightarrow 1$. The strong localization threshold is then reached when L_0 reaches the localization length $\xi_{loc} \simeq \ell_s \exp(\pi k\ell_{tr}/2)$.
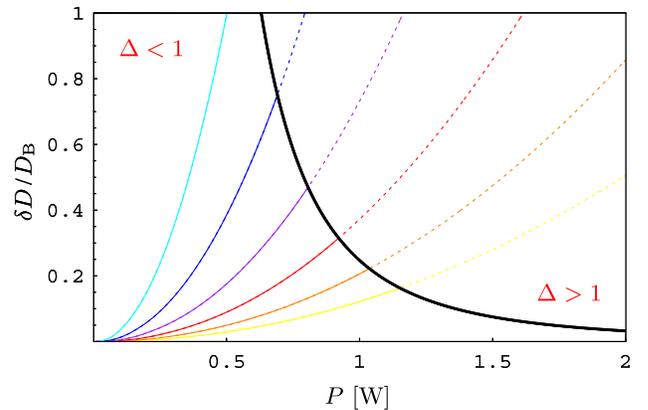

FIG. 2 (color online). Weak localization corrections δD relative to the Boltzmann diffusion constant D_B at detuning $\delta = 10^6\Gamma$, Eq. (7), as a function of laser power P for different atomic matter wave numbers (from left to right) $k\zeta = 1.25$, 1.5, 1.75, 2.0, 2.25, 2.5. The speckle size is $L = 2$ cm, the aperture angle $\alpha = 0.1$. For each value of $k\zeta$, the weak scattering condition $\Delta < 1$ is valid to the left of the thick black line (solid curves). Already for moderate laser power, weak localization corrections of observable size are predicted.
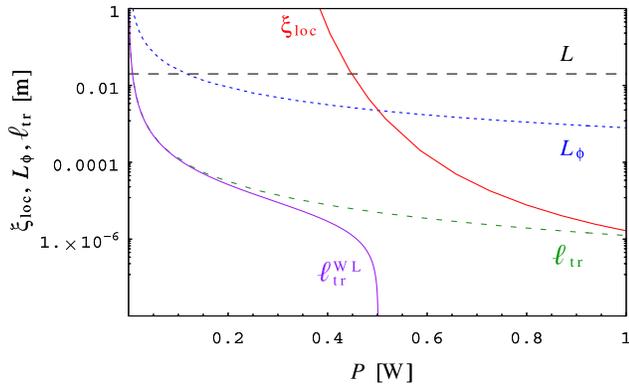
P H Y S I C A L R E V I E W L E T T E R S

FIG. 3 (color online). Logarithmic plot of the 2D localization length ξ_{loc} and of the phase coherence length L_ϕ as a function of the laser power P for $k\zeta = 1.25$ and $\delta = 10^6\Gamma$. The system size is fixed at $L = 2$ cm, the aperture angle at $\alpha = 0.1$. ξ_{loc} and L_ϕ cross at the strong localization threshold, which is reached for $P = 0.5$ W, where the corrected transport mean free path $\ell_{tr}^{WL} = 2m(D_B - \delta D)/\hbar k$ vanishes.

Because of its exponential growth, the localization length becomes very large when $k\ell_s$ increases. Observing strong localization requires fully coherent scattering in a sufficiently large speckle field. We therefore have to check whether these requirements can be met with reasonable laser power and atom temperatures. In Fig. 3, we have plotted ξ_{loc} and L_ϕ as a function of P for $k\zeta = 1.25$, $L = 2$ cm, $\alpha = 0.1$, and $\delta = 10^6\Gamma$. The two curves cross when $P = 0.5$ W. At this point, we find for Rb^{87} atoms ($\lambda_L = 2\pi/k_L = 0.78$ μm) $\xi_{loc} = 2$ mm, $\ell_{tr} = 5$ μm, and $\ell_s = 0.92$ μm. This places the strong localization threshold at $k\ell_s \approx 0.93$ and $\eta \approx 0.77$, meaning that the atoms have an energy slightly above the speckle fluctuations, $E \approx 1.02\bar{V}$. The atomic temperature is then $T = 2.8$ nK which is experimentally accessible [26]. The corresponding de Broglie wavelength is $\lambda_{dB} = 8\lambda_L \approx 6$ μm.

To summarize.—Making use of quantum transport theory, we have studied the dynamics of ultracold atoms in 2D speckle potentials. Starting from the microscopic potential correlation function, we have calculated the elastic scattering mean free path and the classical diffusion constant. Weak localization corrections are shown to be of measurable size for realistic laser power and atom temperature. We have found that the threshold to the strong localization regime is experimentally accessible and therefore worth further numerical and experimental investigation. We have given an estimate for the localization length and provided a set of reasonable experimental parameters that we hope will facilitate the realization in the laboratory.

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