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Bogoliubov theory on the disordered lattice

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Abstract. Quantum fluctuations of Bose-Einstein condensates trapped in disordered lattices are studied by inhomogeneous Bogoliubov theory. Weak-disorder perturbation theory is applied to compute the elastic scattering rate as well as the renormalized speed of sound in lattices of arbitrary dimensionality. Furthermore, analytical results for the condensate depletion are presented, which are in good agreement with numerical data.

1 Introduction

In contrast to dense quantum fluids like liquid Helium, dilute gases of ultracold bosons can be condensed into rather pure Bose-Einstein condensates. But due to interactions, there is always a fraction of non-condensed particles, even at zero temperature [1–3]. This fraction can be enlarged by loading the atoms into the deep wells of optical lattices [4], until finally the celebrated quantum phase transition from superfluid to Mott insulator is reached [5–7]. A somewhat different quantum phase transition from the superfluid to the Bose-glass is driven by applying instead a potential that varies randomly in space [8–11]. Generally, the competition between kinetic energy, interaction and disorder makes this so-called dirty boson problem very rich in its phenomenology, and different parameter regimes require different approaches [12–14].

Here, we investigate some disorder-related properties of the deep superfluid phase preponderant at high filling in the Bose-Hubbard model, for which Bogoliubov theory provides the right framework [1,15]. Bogoliubov theory has been used extensively for disordered Bose gases [16–18]. Here, we transfer our inhomogeneous Bogoliubov theory [19] from the continuum onto the lattice, which amounts mostly (but not entirely) to replacing the free-space dispersion by the lattice dispersion. The following Sects. 2 and 3 describe the model and its Bogoliubov theory in greater detail. As a first application, we compute the effective dispersion relation with scattering rates and speed-of-sound corrections in Sect. 4.

It is an essential feature of our approach that quantum fluctuations are counted from the underlying condensate that is itself already deformed by the external

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potential. In such a setting, it becomes quite a challenge to distinguish the condensed from the non-condensed component [20–23]. Recent Quantum Monte Carlo studies have addressed this question both for weak lattices [24] and harmonic traps [25]. In Sect. 5, we use the approach described in [26] to determine the disorder-induced quantum depletion in the lattice. We reach excellent agreement with numerical data obtained by Singh and Rokhsar [27] within an explicit Bogoliubov diagonalization, and thus validate our analytical formulation. Section 6 concludes.

2 Bosons on a random lattice

2.1 Disordered Bose-Hubbard model

We consider bosonic atoms destroyed (created) by field operators $\hat{c}_{\boldsymbol{x}}$ ($\hat{c}_{\boldsymbol{x}}^\dagger$) at the sites $\boldsymbol{x} = \sum_{j=1}^{d} n_j \boldsymbol{a}_j$, $n_j \in \mathbb{Z}$, of a simple Bravais lattice with basis vectors $\boldsymbol{a}_j$, $1 \leq j \leq d$, fulfilling the canonical commutation relations $\left[\hat{c}_{\boldsymbol{x}}, \hat{c}_{\boldsymbol{x}'}^\dagger\right] = \delta_{\boldsymbol{x}\boldsymbol{x}'}$, $\left[\hat{c}_{\boldsymbol{x}}, \hat{c}_{\boldsymbol{x}'}\right] = 0$, and $\left[\hat{c}_{\boldsymbol{x}}^\dagger, \hat{c}_{\boldsymbol{x}'}^\dagger\right] = 0$. The system is governed by the grand-canonical single-band Bose-Hubbard Hamiltonian

$$\hat{E} = \sum_{\boldsymbol{x}} \left\{ \sum_{\boldsymbol{x}'} \hat{c}_{\boldsymbol{x}}^\dagger T_{\boldsymbol{x}\boldsymbol{x}'} \hat{c}_{\boldsymbol{x}'} + (V_{\boldsymbol{x}} - \mu) \hat{c}_{\boldsymbol{x}}^\dagger \hat{c}_{\boldsymbol{x}} + \frac{U}{2} \hat{c}_{\boldsymbol{x}}^\dagger \hat{c}_{\boldsymbol{x}}^\dagger \hat{c}_{\boldsymbol{x}} \hat{c}_{\boldsymbol{x}} \right\}. \tag{1}$$

The hopping matrix $T_{\boldsymbol{x}\boldsymbol{x}'} = -\sum_{j=1}^{d} J_j (\delta_{\boldsymbol{x},\boldsymbol{x}'+\boldsymbol{a}_j} + \delta_{\boldsymbol{x},\boldsymbol{x}'-\boldsymbol{a}_j} - 2\delta_{\boldsymbol{x},\boldsymbol{x}'})$ contains the tunneling amplitudes J_j along direction j between nearest-neighbor Wannier orbitals in the lowest band. To ease notations, we consider in the following a simple cubic lattice with edges of length $L_j = \mathcal{N}_j a$ and a total number of sites $\mathcal{N} = \prod_j \mathcal{N}_j$; the generalization to other geometries is straightforward. The chemical potential μ controls the particle number, and $U > 0$ is the onsite repulsion. Equation (1) describes the simplest version of the disordered Bose-Hubbard model, where only the on-site energy $V_{\boldsymbol{x}}$ is random [8,12,27]. More generally, randomness can also occur in hopping and interaction [21,22].

Without the external potential, the system is translation invariant, so it is appropriate to use the Fourier representation $\hat{c}_{\boldsymbol{k}} = \mathcal{N}^{-\frac{1}{2}} \sum_{\boldsymbol{x}} \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{x}\right) \hat{c}_{\boldsymbol{x}}$, with allowed wave vectors $\boldsymbol{k}$ in the first Brillouin zone such that $\sum_{\boldsymbol{k}} \exp[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')] = \mathcal{N}\delta_{\boldsymbol{x}\boldsymbol{x}'}$. The kinetic energy operator becomes diagonal, and its matrix elements $T_{\boldsymbol{k}\boldsymbol{k}'} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\varepsilon_{\boldsymbol{k}}^0$ define the clean lattice dispersion

$$\varepsilon_{\boldsymbol{k}}^0 = 2J \sum_{j=1}^{d} [1 - \cos(ak_j)]. \tag{2}$$

In the continuous limit $a \to 0$, the hopping matrix turns into the continuous Laplacian, and $2Ja^2 \to \hbar^2/m$, such that $\varepsilon_{\boldsymbol{k}}^0 \to \hbar^2 k^2/2m$. In the remainder of this contribution, we will be concerned with the inverse mapping, namely to transfer the continuum results of [19,26] to the lattice case.

2.2 Statistical properties of the disorder potential

Hitherto, cold-atom experimentalists have explored three ways to realize random lattices: (i) Sampling a continuous potential $V(\boldsymbol{r})$, such as optical speckle, at the lattice positions [21,22]; (ii) Trapping impurity atoms at or near the primary lattice

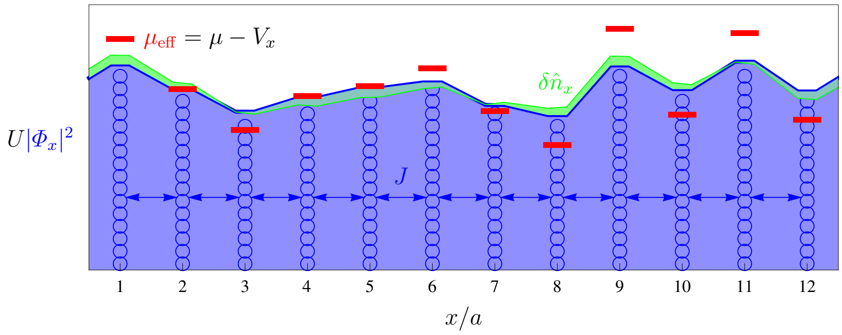


Fig. 1. Schematic view of a disordered lattice hosting an extended, but deformed condensate. In the atomic limit $J = 0$, the condensate density (blue) would fill the wells up to $(\mu - V_{\boldsymbol{x}})/U$. Finite hopping J (in this plot $J = Un_{\rm c}$) couples adjacent sites and smoothens the profile. Section 3 and following discuss the quantum fluctuations $\delta\hat{n}$ (and $\delta\hat{\varphi}$), here cartooned in green.

sites [28,29]; (iii) Creating a quasi-random distribution by adding an incommensurate lattice to the primary lattice [30].

The random potential $V_{\boldsymbol{x}}$ is characterized by its moments $\overline{V_{\boldsymbol{x}}}$, $\overline{V_{\boldsymbol{x}} V_{\boldsymbol{x}'}}$, etc. Without loss of generality, we can set $\overline{V_{\boldsymbol{x}}} = 0$ by readjusting the zero of energy. Concerning the covariance, one is capable of realizing, in all of the above-mentioned cases, spatially uncorrelated disorder such that $\overline{V_{\boldsymbol{x}} V_{\boldsymbol{x}'}} = V^2 \delta_{\boldsymbol{x}\boldsymbol{x}'}$, or

$$\overline{V_{\boldsymbol{k}} V_{-\boldsymbol{k}'}} = V^2 \mathcal{N}^{-1} \delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{3}$$

for the Fourier components $V_{\boldsymbol{k}} = \mathcal{N}^{-1} \sum_{\boldsymbol{x}} \exp(-i\boldsymbol{k} \cdot \boldsymbol{x}) V_{\boldsymbol{x}}$. In the speckle case (i), this is valid if the spatial correlation length is not larger than the lattice constant; for the Bernoulli disorder (ii) [11,28], this is true by construction, and in a quasi-random system (iii) this requires the incommensurability to be large enough so that no repetitions occur in the finite size under study [31].

2.3 Condensate deformation on the mean-field level

When the discreteness of the particle number at a given site is negligible, i.e., for high particle numbers per site, $n \gg U/J$, a mean-field description for the extended condensate is a good starting point (see the schematic representation in Fig. 1). Replacing the operators $\hat{c}_{\boldsymbol{x}}^{(\dagger)}$ by their ground-state expectation value $\Phi_{\boldsymbol{x}}$ (which can be chosen real) and minimizing (1) yields the Gross-Pitaevskii equation, which we write in the Fourier representation:

$$\mu\phi_{\boldsymbol{k}} = \varepsilon^0_{\boldsymbol{k}} \phi_{\boldsymbol{k}} + \sum_{\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'} \phi_{\boldsymbol{k}'} + \frac{U}{\mathcal{N}} \sum_{\boldsymbol{k}'\boldsymbol{k}''} \phi_{\boldsymbol{k}-\boldsymbol{k}'} \phi_{\boldsymbol{k}'-\boldsymbol{k}''} \phi_{\boldsymbol{k}''}. \tag{4}$$

If the external potential is weak, the homogeneous solution $\phi^{(0)}_{\boldsymbol{k}} = N_{\rm c}^{1/2} \delta_{\boldsymbol{k}0}$ is only slightly perturbed, and one can compute the deformed condensate amplitude $\phi_{\boldsymbol{k}} = \phi^{(0)}_{\boldsymbol{k}} + \phi^{(1)}_{\boldsymbol{k}} + \dots$ perturbatively in powers of V [12]. At fixed number of condensed particles, $\sum_{\boldsymbol{k}} |\phi_{\boldsymbol{k}}|^2 = N_{\rm c}$, also the chemical potential has to be expanded. As a result, the condensate deformation reads $\phi^{(1)}_{\boldsymbol{k}} = -N_{\rm c}^{1/2} \tilde{V}_{\boldsymbol{k}}$ to first order in the reduced potential matrix element

$$\tilde{V}_{\boldsymbol{k}} = \frac{1 - \delta_{\boldsymbol{k}0}}{\varepsilon^0_{\boldsymbol{k}} + 2Un_{\rm c}} V_{\boldsymbol{k}}. \tag{5}$$

This expression matches Eq. (9) in [19] [or Eq. (15) in [12]], after substituting gn by $Un_{\mathrm{c}} = UN_{\mathrm{c}}/\mathcal{N}$, and reading $\varepsilon^0_{\boldsymbol{k}}$ as the lattice dispersion (2). Also all higher-order terms agree with their respective expressions in [19], as do derived quantities like the inverse condensate amplitude and densities.

In the limit $J \ll Un_{\mathrm{c}}$ of weak coupling between sites (which requires large occupation such that still $Jn_{\mathrm{c}} \gg U$), the bare kinetic energy $\varepsilon^0_{\boldsymbol{k}}$ in the denominator of the reduced potential (5) is always negligible compared to $2Un_{\mathrm{c}}$. Thus, the condensate faithfully follows the potential, to all orders, resulting in a Thomas-Fermi profile and a screened disorder landscape [12]. In the strong-coupling limit $J \gg Un_{\mathrm{c}}$, kinetic and interaction energy become comparable at a characteristic wave vector equal to the inverse of the healing length, $\xi = a\sqrt{J/(Un_{\mathrm{c}})} \gg a$. Whereas potential fluctuations on large scales $k^{-1} \gg \xi$ are still followed by the condensate, fluctuations on small scales $k^{-1} \ll \xi$ cost too much kinetic energy, and result only in a smoothed imprint [32].

The deformed condensate consists of particles with the (ensemble-averaged) momentum distribution [26]

$$n_{\mathrm{c}\boldsymbol{k}} = \overline{|\phi^2_{\boldsymbol{k}}|} = N_{\mathrm{c}}\left[(1 - V_2)\delta_{\boldsymbol{k}0} + \overline{|\tilde{V}^2_{\boldsymbol{k}}|}\right]; \tag{6}$$

the second equality holds to order V^2 in the disorder strength. Here, $V_2 = \sum_{\boldsymbol{k}} \overline{|\tilde{V}^2_{\boldsymbol{k}}|}$ is the fraction of condensed particles with $\boldsymbol{k} \neq 0$, sometimes referred to as glassy fraction or Edwards-Anderson order parameter [33]. In the weak-coupling limit $J \ll Un_{\mathrm{c}}$ and for uncorrelated disorder (3), this fraction is $V_2 = v^2/4$, where $v^2 = (V/Un_{\mathrm{c}})^2$ is the disorder variance in units of the chemical potential of the homogeneous lattice gas.

3 Inhomogeneous Bogoliubov Hamiltonian on the lattice

In this section, we set up the inhomogeneous Bogoliubov Hamiltonian for quantum fluctuations $\delta\hat{c}_{\boldsymbol{x}}$ around the (deformed) mean-field condensate $\Phi_{\boldsymbol{x}}$ on a disordered lattice. For the reasons given in [19], a well-defined theory ensues if one expresses the fluctuations via density and phase,

$$\delta\hat{c}_{\boldsymbol{x}} = \hat{c}_{\boldsymbol{x}} - \Phi_{\boldsymbol{x}} = \frac{\delta\hat{n}_{\boldsymbol{x}}}{2\Phi_{\boldsymbol{x}}} + \mathrm{i}\Phi_{\boldsymbol{x}}\delta\hat{\varphi}_{\boldsymbol{x}}. \tag{7}$$

Inserting (7) into (1), and collecting all second-order terms,[1] one arrives at a fluctuation Hamiltonian of the form

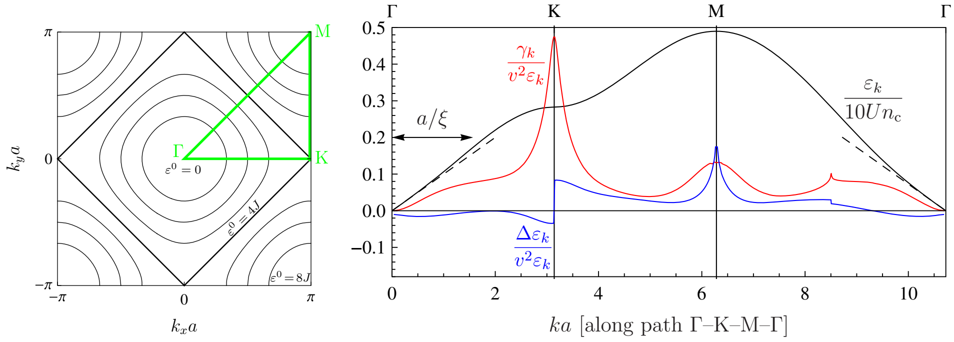
$$\hat{H} = \frac{1}{2}\sum_{\boldsymbol{k}\boldsymbol{k}'}\left[n_{\mathrm{c}}\delta\hat{\varphi}^\dagger_{\boldsymbol{k}} S_{\boldsymbol{k}\boldsymbol{k}'}\delta\hat{\varphi}_{\boldsymbol{k}'} + \frac{1}{4n_{\mathrm{c}}}\delta\hat{n}^\dagger_{\boldsymbol{k}} R_{\boldsymbol{k}\boldsymbol{k}'}\delta\hat{n}_{\boldsymbol{k}'}\right]. \tag{8}$$

Crossed terms have canceled, up to constants from commutators, and the coupling matrices between different Fourier modes are found to be

$$S_{\boldsymbol{k}\boldsymbol{k}'} = \frac{2}{N_{\mathrm{c}}}\sum_{\boldsymbol{p}} \phi_{\boldsymbol{k}-\boldsymbol{p}}\phi_{\boldsymbol{p}-\boldsymbol{k}'}(\varepsilon^0_{\boldsymbol{p}} - \varepsilon^0_{\boldsymbol{k}-\boldsymbol{p}}), \tag{9}$$

$$R_{\boldsymbol{k}\boldsymbol{k}'} = \frac{2}{N_{\mathrm{c}}}\sum_{\boldsymbol{p},\boldsymbol{q}} \check{\phi}_{\boldsymbol{k}-\boldsymbol{p}}\check{\phi}_{\boldsymbol{p}-\boldsymbol{k}'-\boldsymbol{q}}\left[(\varepsilon^0_{\boldsymbol{p}} - \mu)\delta_{\boldsymbol{q}0} + V_{\boldsymbol{q}}\right] + 6Un_{\mathrm{c}}\delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{10}$$

[1] First-order terms vanish because of (4). Higher-order terms describe interactions among the excitations that remain negligible for very low temperatures and dilute gases.



(a) (b)

Fig. 2. (a) Equipotential lines of the clean and Bogoliubov lattice dispersions, (2) and (12), respectively, in the first Brillouin zone of the 2D lattice. (b) Plots of several quantities for $J = 0.5Un_c$ along the closed path connecting Γ, K, and M in (a): Clean Bogoliubov dispersion $\varepsilon_{\boldsymbol{k}}$, Eq. (12), in units of $10Un_c$ (full black line), together with the phonon dispersion $\varepsilon_{\boldsymbol{k}} = \hbar c k$ for $k\xi \ll 1$ (dashed black). Colored curves show the disorder corrections calculated in Sect. 4: relative scattering rate $\gamma_{\boldsymbol{k}}/\varepsilon_{\boldsymbol{k}}$ (red); relative dispersion shift $\Delta\varepsilon_{\boldsymbol{k}}/\varepsilon_{\boldsymbol{k}}$ (blue), both in units of $v^2 = V^2/(Un_c)^2$.

Here, $\check{\phi}_{\boldsymbol{k}} = [n_c/\phi]_{\boldsymbol{k}}$ are Fourier components of the inverse condensate amplitude. Equations (9) and (10) are the lattice equivalents of (29) and (30) in [19]. There, we used the identity $\check{\phi}\nabla\phi = -\phi\nabla\check{\phi}$ together with (4) to express $R_{\boldsymbol{k}\boldsymbol{k}'}$ as a quadratic functional of $\check{\phi}_{\boldsymbol{k}}$ without appearance of $V_{\boldsymbol{q}}$ nor μ, which turns out impossible on the lattice.

3.1 Homogeneous case

In absence of the potential $V_{\boldsymbol{x}}$, the condensate is homogeneous, and the coupling matrices reduce to $S^{(0)}_{\boldsymbol{k}\boldsymbol{k}'} = 2\varepsilon^0_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'}$ and $R^{(0)}_{\boldsymbol{k}\boldsymbol{k}'} = 2[\varepsilon^0_{\boldsymbol{k}} + 2Un_c]\delta_{\boldsymbol{k}\boldsymbol{k}'}$. The Hamiltonian becomes diagonal, $\hat{H}^{(0)} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}}\hat{\gamma}^\dagger_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}}$, after a canonical transformation to Bogoliubov quasiparticles ("bogolons") such that $[\gamma_{\boldsymbol{k}'}, \gamma^\dagger_{\boldsymbol{k}}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. This is achieved by

$$\begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix} = A_{\boldsymbol{k}} \begin{pmatrix} \mathrm{i}\sqrt{n_c}\,\delta\hat{\varphi}_{\boldsymbol{k}} \\ \delta\hat{n}_{\boldsymbol{k}}/(2\sqrt{n_c}) \end{pmatrix}, \qquad A_{\boldsymbol{k}} = \begin{pmatrix} a_{\boldsymbol{k}} & a^{-1}_{\boldsymbol{k}} \\ -a_{\boldsymbol{k}} & a^{-1}_{\boldsymbol{k}} \end{pmatrix} \tag{11}$$

with parameter $a_{\boldsymbol{k}} = (\varepsilon^0_{\boldsymbol{k}}/\varepsilon_{\boldsymbol{k}})^{1/2}$ in each $\boldsymbol{k}$-sector. The resulting Bogoliubov dispersion

$$\varepsilon_{\boldsymbol{k}} = [\varepsilon^0_{\boldsymbol{k}}(\varepsilon^0_{\boldsymbol{k}} + 2Un_c)]^{1/2} \tag{12}$$

is plotted in Fig. 2(b) along a representative path in the Brillouin zone of a 2D simple cubic lattice shown in Fig. 2(a). For the chosen parameters, the healing length $\xi = 2a/\pi$ separates the sound-wave regime $k\xi \ll 1$ with linear dispersion from the particle-like excitation regime $k\xi \gg 1$ with parabolic dispersion.

3.2 Bogoliubov scattering vertex

For any given realization of the external potential $V_{\boldsymbol{x}}$, and thus a given realization of the coupling matrices (9) and (10), one can diagonalize the quadratic Hamiltonian (8).

Analytical calculations for weak disorder, however, are best done in the bogolon basis (11)–with density (and phase) fluctuations counted from the deformed condensate. The price for keeping a plane-wave basis is the appearance of an impurity-scattering term in the Hamiltonian, which can be treated using standard perturbation theory.

We separate the homogeneous contributions from the coupling matrices by defining $R^{(V)}_{\boldsymbol{kk}'} = R_{\boldsymbol{kk}'} - R^{(0)}_{\boldsymbol{kk}'}$, as well as $S^{(V)}_{\boldsymbol{kk}'} = S_{\boldsymbol{kk}'} - S^{(0)}_{\boldsymbol{kk}'}$. By the canonical transformation (11) to Nambu pseudo-spinors $\hat{\Gamma}^\dagger_{\boldsymbol{k}} = (\hat{\gamma}^\dagger_{\boldsymbol{k}}, \hat{\gamma}_{-\boldsymbol{k}})/\sqrt{2}$, the fluctuation Hamiltonian (8) is brought into impurity-scattering form [19]:

$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \hat{\Gamma}^\dagger_{\boldsymbol{k}} \hat{\Gamma}_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Gamma}^\dagger_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{kk}'} \hat{\Gamma}_{\boldsymbol{k}'}. \tag{13}$$

The bogolon scattering vertex[2] $\mathcal{V}_{\boldsymbol{kk}'} = (A^{-1}_{\boldsymbol{k}})^\dagger \mathrm{diag}\big(S^{(V)}_{\boldsymbol{kk}'}, R^{(V)}_{\boldsymbol{kk}'}\big) A^{-1}_{\boldsymbol{k}'}$ is a function of the condensate ground-state configuration via (9) and (10), and can be expanded in powers of the bare potential using the perturbation theory of Sect. 2.3. Thus, our approach goes beyond earlier Bogoliubov theories on disordered lattices [12,27] inasmuch our Hamiltonian is useful even for purely analytical calculations. To first order,

$$R^{(1)}_{\boldsymbol{kk}'} = 2\tilde{V}_{\boldsymbol{k}-\boldsymbol{k}'} \left[\varepsilon^0_{\boldsymbol{k}-\boldsymbol{k}'} + \varepsilon^0_{\boldsymbol{k}} + \varepsilon^0_{\boldsymbol{k}'}\right], \tag{14}$$

$$S^{(1)}_{\boldsymbol{kk}'} = 2\tilde{V}_{\boldsymbol{k}-\boldsymbol{k}'} \left[\varepsilon^0_{\boldsymbol{k}-\boldsymbol{k}'} - \varepsilon^0_{\boldsymbol{k}} - \varepsilon^0_{\boldsymbol{k}'}\right]. \tag{15}$$

If one is interested in results to second order at most, at this order only disorder-averaged, diagonal terms are needed:

$$\overline{S^{(2)}_{\boldsymbol{kk}'}} = 2\delta_{\boldsymbol{kk}'} \sum_{\boldsymbol{q}} \overline{\big|\tilde{V}_{\boldsymbol{q}}\big|^2} \left[\varepsilon^0_{\boldsymbol{k}-\boldsymbol{q}} - \varepsilon^0_{\boldsymbol{k}} - \varepsilon^0_{\boldsymbol{q}}\right], \tag{16}$$

$$\overline{R^{(2)}_{\boldsymbol{kk}'}} = 2\delta_{\boldsymbol{kk}'} \sum_{\boldsymbol{q}} \overline{\big|\tilde{V}_{\boldsymbol{q}}\big|^2} \left[\varepsilon^0_{\boldsymbol{k}-\boldsymbol{q}} + 3\varepsilon^0_{\boldsymbol{k}} + 3\varepsilon^0_{\boldsymbol{q}}\right]. \tag{17}$$

Contrary to the continuous-limit case [19], where $S_{\boldsymbol{kk}'}$ of Eq. (28) is proportional to $n_{\mathrm{c}\boldsymbol{k}-\boldsymbol{k}'}$, here its diagonal elements have no reason to vanish.

4 Effective medium theory

One of the first things one may want to know is how the disorder modifies the excitation spectrum. Clearly, randomness limits the life-time of plane-wave excitations, affects the speed of sound, changes the density of states, and causes localization [19,34–37]. All these effects can be assessed by applying diagrammatic perturbation theory to the Hamiltonian (13), for instance via a Green's function formalism. The Nambu-Green function of the clean Hamiltonian is $\mathcal{G}_{0\boldsymbol{k}}(z) = \mathrm{diag}\big(G_{0\boldsymbol{k}}(z), G_{0\boldsymbol{k}}(-z)\big)$, with $G_{0\boldsymbol{k}}(z) = [z - \varepsilon_{\boldsymbol{k}}]^{-1}$, and the full Green function obeys the recursive equation $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}$. The ensemble-averaged Green function $\overline{\mathcal{G}} = \mathcal{G}_0 + \mathcal{G}_0 \Sigma \overline{\mathcal{G}}$ describes the propagation of bogolons through an effective medium, whose effect is to renormalize the dispersion:

$$\hbar\omega_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} + \Sigma_{11}(\hbar\omega, \boldsymbol{k}) = \varepsilon_{\boldsymbol{k}} + \Delta\varepsilon_{\boldsymbol{k}} - \mathrm{i}\gamma_{\boldsymbol{k}}/2. \tag{18}$$

[2] Equation (31) in [19] contains a misprint with two factors of 1/2 too much; the following expressions are not affected.

Σ_{11} designates the first block of the Nambu self-energy. To second order in the bare potential, one finds two types of terms, $\Sigma^{(2)} = \overline{\mathcal{V}^{(1)}\mathcal{G}_0\mathcal{V}^{(1)}} + \overline{\mathcal{V}^{(2)}}$, resulting from the perturbative expansion of the bogolon vertex.

Let us first discuss the elastic scattering rate $\gamma_{\boldsymbol{k}}$. To lowest order, known as the Born approximation, it is given by Fermi's Golden Rule via a momentum integral over the energy shell:

$$\gamma_{\boldsymbol{k}} = \frac{\pi}{8} \sum_{\boldsymbol{p}} \overline{\left| a_{\boldsymbol{k}} a_{\boldsymbol{p}} R^{(1)}_{\boldsymbol{kp}} + a_{\boldsymbol{k}}^{-1} a_{\boldsymbol{p}}^{-1} S^{(1)}_{\boldsymbol{kp}} \right|^2} \delta(\varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{p}}). \tag{19}$$

Due to the rather complicated form of the energy shells, even for a simple cubic lattice, the integral has to be evaluated numerically.[3] In Fig. 2(b), the relative scattering rate $\gamma_{\boldsymbol{k}}/\varepsilon_{\boldsymbol{k}}$ in units of v^2 is shown across the Brillouin zone. Just as in the continuous limit, the scattering of low-energy excitations near the Γ point is suppressed, resulting in long-lived, soft modes. Strong scattering occurs near the symmetry point K at the Brillouin zone boundary.

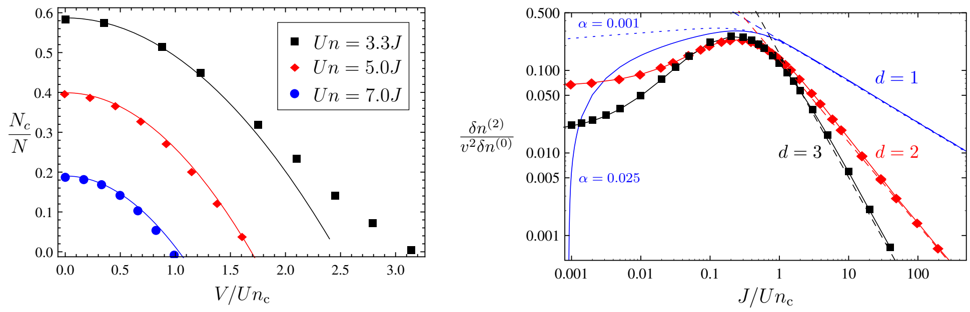
The real part of the self-energy in (18) shifts the energy of excitations. In Fig. 2(b), the shift $\Delta\varepsilon_{\boldsymbol{k}}/\varepsilon_{\boldsymbol{k}}$ relative to the clean dispersion is plotted in units of v^2. The shift $\Delta\varepsilon_{\boldsymbol{k}}$ is a principle-value integral over a kernel with a simple pole at the energy shell [19]. As shown in Fig. 2(a), the energy shell changes its topology at $\varepsilon = [4J(4J+Un_{\mathrm{c}})]^{1/2}$, which causes jumps in the shift of the dispersion $\Delta\varepsilon_{\boldsymbol{k}}$. For the soft modes, $\Delta\varepsilon_{\boldsymbol{k}}$ results in a changed speed of sound, $\overline{c} = c + \Delta\overline{c}$, which we observe to be negative in the 2D example of Fig. 2(b). In the limit $J/Un_{\mathrm{c}} = \xi^2/a^2 \gg 1$, the lattice constant a drops out, and we recover the asymptotics for uncorrelated disorder in the continuum, $\Delta\overline{c}/c \propto v_\delta^2 = v^2(a/\xi)^d$, where the proportionality factor is negative in 1D, positive in 3D and vanishes in 2D [19]. The increase in 3D, $\Delta\overline{c}/c = 5v_\delta^2/(48\sqrt{2}\pi)$, coincides with the established result [17,38,39] for uncorrelated disorder. However, as noted in [19], an increase of c with disorder is untypical. Generically, one expects the disorder to reduce the speed of sound, and consequently to increase the low-energy density of states. And really, in the opposite, weak-coupling limit $J \ll Un_{\mathrm{c}}$, the shift is negative and reduces to the value $\Delta\overline{c}/c = -0.375v^2, -0.215v^2, -0.177v^2$ in dimensions 1, 2, and 3, respectively. This is very similar to the hydrodynamic regime of the continuum [19,35], but with slightly different constants.

5 Condensate depletion

Due to the repulsive interaction between atoms, there is always a certain fraction of particles outside of the condensate. In a weakly interacting gas, this fraction can be estimated using Bogoliubov theory. Let us first discuss the homogeneous lattice case. Condensation occurs in the state with quasimomentum $\boldsymbol{k} = 0$, and the non-condensed particles have a momentum distribution $\delta n^{(0)}_{\boldsymbol{k}} = \langle \delta\hat{c}^\dagger_{\boldsymbol{k}} \delta\hat{c}_{\boldsymbol{k}} \rangle = (a_{\boldsymbol{k}} - a_{\boldsymbol{k}}^{-1})^2/4$, as follows from (7) and (11) in the Bogoliubov vacuum at $T = 0$. The total density of non-condensed particles (number per lattice site) is

$$\delta n^{(0)} = \mathcal{N}^{-1} \sum_{\boldsymbol{k}} \frac{(\varepsilon_{\boldsymbol{k}} - \varepsilon^0_{\boldsymbol{k}})^2}{4\varepsilon_{\boldsymbol{k}}\varepsilon^0_{\boldsymbol{k}}}. \tag{20}$$

[3] We parametrize the integral by the polar (and azimuthal, in $d = 3$) angle, determine the modulus of $\boldsymbol{p}$ such that $\varepsilon_{\boldsymbol{p}} = \varepsilon_{\boldsymbol{k}}$, and evaluate the integrand there. This has to be multiplied with the surface element and the Jacobian $[\partial\varepsilon_{\boldsymbol{p}}/\partial p_n]^{-1}$, where p_n is the component normal to the energy shell. For high energies, we take the corner M as origin.



(a) (b)

Fig. 3. (a) 2D condensate fraction N_c/N as function of the disorder strength $v = V/Un_c$ for different values of Un_c/J. The data points are taken from Fig. 4 in [27], the result of an exact Bogoliubov diagonalization. The lines show our results for the condensate fraction resulting from the combined clean and potential depletion $\delta n = \delta n^{(0)} + \delta n^{(2)}$, Eqs. (20) and (21). For not too strong disorder $v = V/Un_c \lesssim 1.5$ the agreement is excellent. (b) Disorder-induced depletion, relative to the clean depletion, in dimensions $d = 1, 2, 3$. The dashed lines show the large-J asymptotics $\delta n^{(2)}/\delta n^{(0)} \approx \beta_d v_\delta^2 = v^2(Un_c/J)^{d/2}$, corresponding to the uncorrelated limit of Ref. [26]. In one dimension, an infrared cutoff $\alpha = k_{IR}\xi$ is needed.

In the continuous limit, $J \gg Un_c$, it shows the well-known scaling proportional to $(Un_c/J)^{d/2} = (a/\xi)^d$. In the weak coupling regime, $J \ll Un_c$, the asymptotics is $\delta n^{(0)} \approx g_d\sqrt{Un_c/J}$, where $g_3 = 0.161$ and $g_2 = 0.227$. In one dimension, an infrared cutoff is needed, because there is no true Bose-Einstein condensation, and $\delta n^{(0)}$ grows logarithmically with the cutoff. In all dimensions, the condensate fraction, or relative number of particles in the condensate, is $N_c(0)/N = 1 - \delta n^{(0)}/n$. Using (20) in the thermodynamic limit, we find excellent agreement with the clean condensate fraction shown in Fig. 4 of Singh and Rokhsar [27], and infer $n = 0.33$ particles per site for their data. Even though the density is quite low, and the system parameters do not verify the prima facie condition $U \ll Jn$ of validity for mean-field theory, at this fractional occupation, the clean system has a superfluid ground state. Nonetheless, we cannot expect Bogoliubov theory to yield accurate predictions as the condensate fraction falls below 50% [4,15].

Next, we study the excitation momentum distribution in the presence of disorder. We can express $\delta n_{\boldsymbol{k}}$, still via (7) and (11), in terms of the condensate profile $\Phi_{\boldsymbol{x}}$ and bogolon ground-state expectation values such as $\langle \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}'} \rangle$; for details see [26]. To second order in the disorder potential, the result can be written

$$\delta n^{(2)}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \tilde{M}^{(2)}_{\boldsymbol{kp}} |\tilde{V}_{\boldsymbol{k}-\boldsymbol{p}}|^2. \tag{21}$$

Now the kernel $\tilde{M}^{(2)}_{\boldsymbol{kp}}$ of Eq. (49) in [26] has to be evaluated using the lattice dispersion (2) and the envelope functions that follow from (9) and (10). Just as in the continuous case of [26], the momentum distribution of excitations induced by the disorder broadens the condensate momentum distribution, Eq. (6).

The resulting, additional quantum depletion of the condensate density caused by the disorder potential, or potential depletion for short, is the sum over the corresponding momentum distribution, $\delta n^{(2)} = \mathcal{N}^{-1} \sum_{\boldsymbol{k}} \delta n^{(2)}_{\boldsymbol{k}}$. In Fig. 3(a), we show results for two dimensions and compare them to data from the exact Bogoliubov diagonalization of Ref. [27]. Within our perturbation theory, we can only estimate the quadratic corrections of order V^2, but this appears to be a rather good approximation even for strong disorder up to $V \approx Un_c$. The good agreement validates our analytical approach to inhomogeneous Bogoliubov theory.

In contrast to [27], our theory is analytical up to the evaluation of the integrals, and we do not need to average over realizations of disorder. This allows us to consider higher dimensions with little additional effort. In Fig. 3(b), the relative potential depletion is shown as function of J/Un_c in dimensions $d = 1, 2, 3$. In all cases, the depletion takes its maximum around $J \approx 0.5Un_c$. The large-J behavior coincides with the limit $\sigma \to 0$ of the continuous case. In accordance with [26], we find $\delta n^{(2)}/\delta n^{(0)} \approx \beta_d v_\delta^2$, with $\beta_1 \approx 0.236$ (slightly cutoff-dependent), $\beta_2 \approx 0.135$, and $\beta_3 \approx 0.160$.

In the weak coupling regime, $J \ll Un_c$, $\delta n^{(2)}$ shows the same scaling proportional to $\sqrt{Un_c/J}$ as $\delta n^{(0)}$, such that the ratio $\delta n^{(2)}/v^2\delta n^{(0)}$ tends to a constant (≈ 0.016 in 3D, and ≈ 0.06 in 2D). In one dimension, however, the weak-coupling behavior is cutoff-dependent: as J/Un_c gets smaller than the cutoff α, the ratio diminishes and it is not possible to obtain a well-defined limit, as becomes evident from the two curves with different cutoffs shown in Fig. 3(b).

To summarize this section, we have shown that the disorder potential does indeed increase the condensate depletion, essentially because it deforms the condensate and creates high-density regions, where the interaction-induced depletion is enhanced. The total depletion can be written [26]

$$\delta n = \delta n^{(0)} \left[1 + v^2 \Delta(J/Un_c)\right], \tag{22}$$

where $\Delta(J/Un_c)$, plotted in 3(b), is always smaller than unity. Thus, the potential depletion for not-too-strong disorder remains a relatively small correction, comparable to the homogeneous depletion, up to $V \sim Un_c$ at least. For a large mismatch between J and Un_c, the condensate becomes even more resilient to the disorder by virtue of smoothing (for $J \gg Un_c$) and screening (for $J \ll Un_c$).

6 Conclusions

We have studied the superfluid phase of the disordered Bose-Hubbard model by Bogoliubov theory, transferring our previous continuum formulation to the lattice case. The key point is to describe the quantum fluctuations around a condensate that is deformed by the random potential. This approach permits purely analytical calculations, at least to second order in the disorder strength. We have determined the renormalized dispersion relation, and find strong scattering at the K point (3D: X and M points), where the free dispersion relation has a saddle point. Otherwise, the excitations stay gapless, and soft modes are long lived, with well-defined corrections to the speed of sound. Furthermore, we have calculated by how much the non-condensed fraction increases due to the disorder. This additional depletion is found to agree with earlier, numerical results by Singh and Rokhsar. We determine its dependence on the system parameters and find that the disorder-induced depletion relative to the clean depletion is greatest when the hopping matches the chemical potential, which should provide useful information for further experimental studies.

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