The Quantum Hall Effect in graphene

Seminar: Electronic properties of graphene
SS 2009

Matthias Bädicker

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- Quantum Hall Effect in graphene
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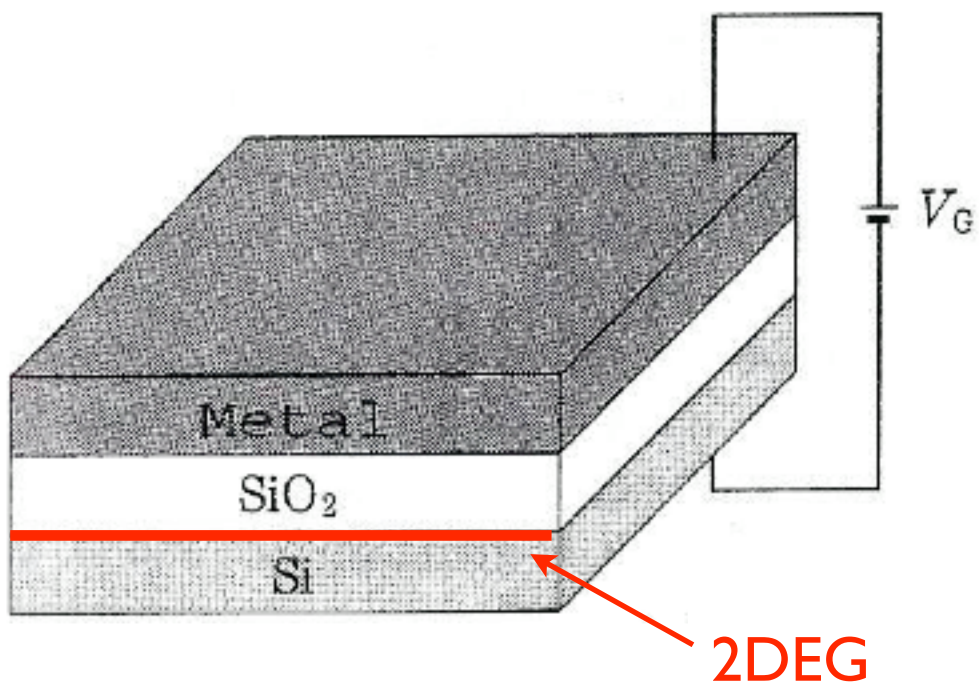
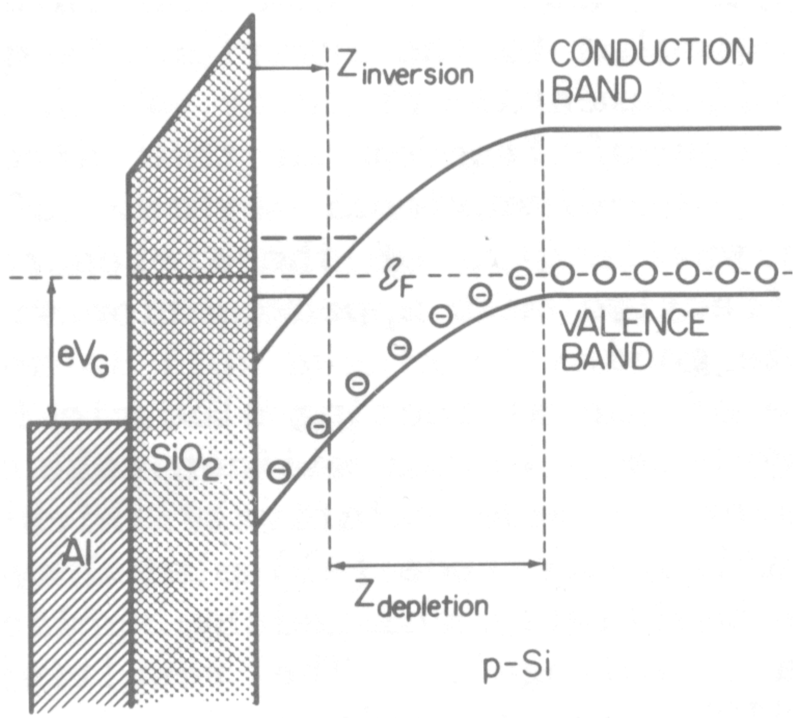
- Summary

2DEG

- 2-dimensional electron gas, builded in a structur metal-oxide-semiconductor





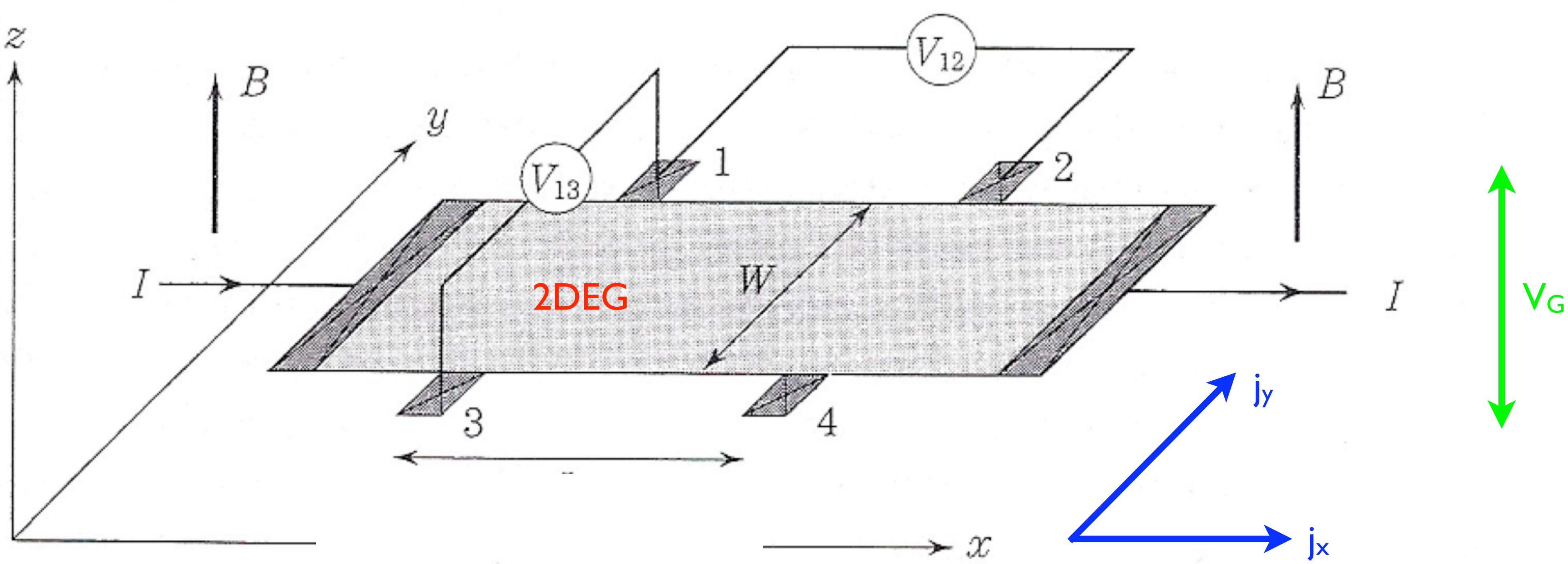
- Variation of the band structure via V_G leads to a very thin layer of quasi-free electrons between the semiconductor and the oxide

- Thickness of the 2DEG: 5-10 nm

Experimental facts

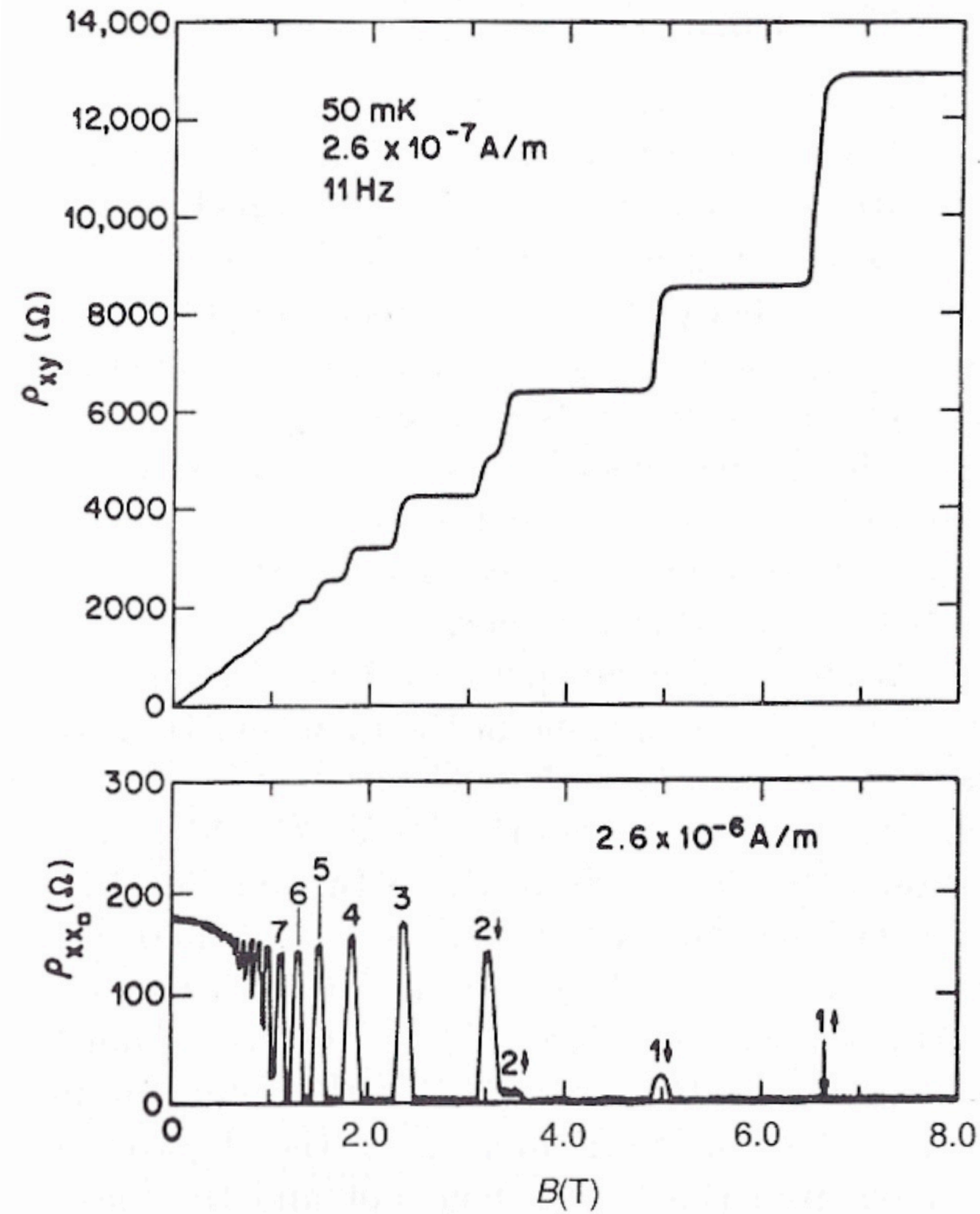
- Current in the 2DEG in x-direction and a magnetic field in z-direction induces a Hall-voltage V_H in the y-direction

$V_{13} = V_H$



$$j_x = \sigma_{xx} E_x$$
$$j_y = \sigma_{xy} E_x$$

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \qquad \rho_{xy} = \frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$


14,000
12,000
10,000
8000
6000
4000
2000
0
50 mK
2.6 x 10^-7 A/m
11 Hz
ρxy (Ω)
300
200
100
0
2.6 x 10^-6 A/m
ρxx□ (Ω)
7
6
5
4
3
2↓
2↑
1↓
1↑
0
2.0
4.0
6.0
8.0
B(T)

The electrons in the 2DEG are described by the Schrödinger-equation:

$$\begin{aligned}\hat{H}\psi(\vec{r}) &= E\psi(\vec{r}) \\ \left(\frac{\vec{p}^2}{2m_e}\right)\psi(\vec{r}) &= E\psi(\vec{r})\end{aligned}$$

Adding a magnetic field, described by the vector potential ***A*** = (- By,0), perpendicular to the x-y-plane leads to:

$$\frac{1}{2m_e}\pi^2\psi(\vec{r}) = \frac{1}{2m_e}\left(\pi_x^2+\pi_y^2\right)\psi(\vec{r}) = E\psi(\vec{r})$$

where $\vec{\pi} = \vec{p} - \frac{-e}{c}\vec{A}$

We also need the following variables:

$$l_B = \sqrt{\frac{\hbar c}{eB}} \qquad \text{magnetic length}$$

$$\omega_c = \frac{eB}{m_e} \qquad \text{cyclotron frequency}$$

Define the following operators:

$$
\begin{aligned}
a &= \alpha\pi_x + \beta\pi_y \\
a^+ &= \alpha^*\pi_x + \beta^*\pi_y
\end{aligned}
$$

These operators should be ladder operators, so they have to fulfil:

$$[a, a^+] = 1$$

So we find the annihilation- and the creation-operators:

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2}}\frac{l_B}{\hbar}(\pi_x - i\pi_y) \\
a^+ &= \frac{1}{\sqrt{2}}\frac{l_B}{\hbar}(\pi_x + i\pi_y)
\end{aligned}
$$

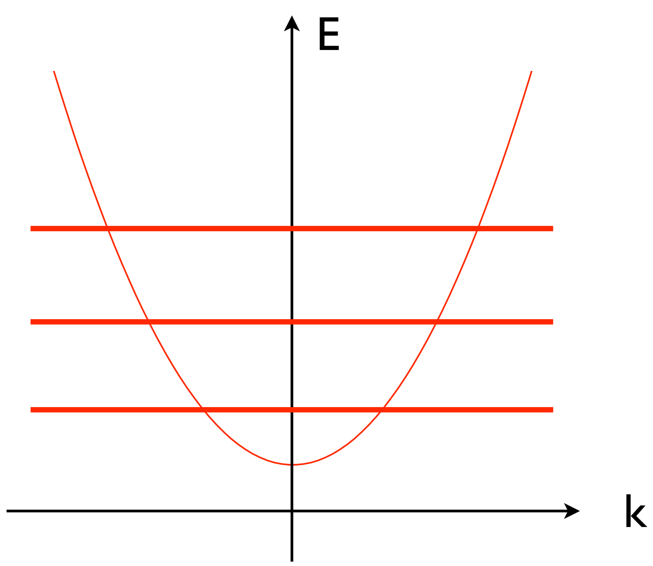
Rewrite the Hamiltonian with this operators:

$$\hat{H} = \hbar\omega_c\left(a^+a + \frac{1}{2}\right)$$

The eigenenergies with *n = 0,1,2,...* are called the **Landau levels**

$$E = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

A magnetic field quantizises the parabolic energy functions in a 2DEG



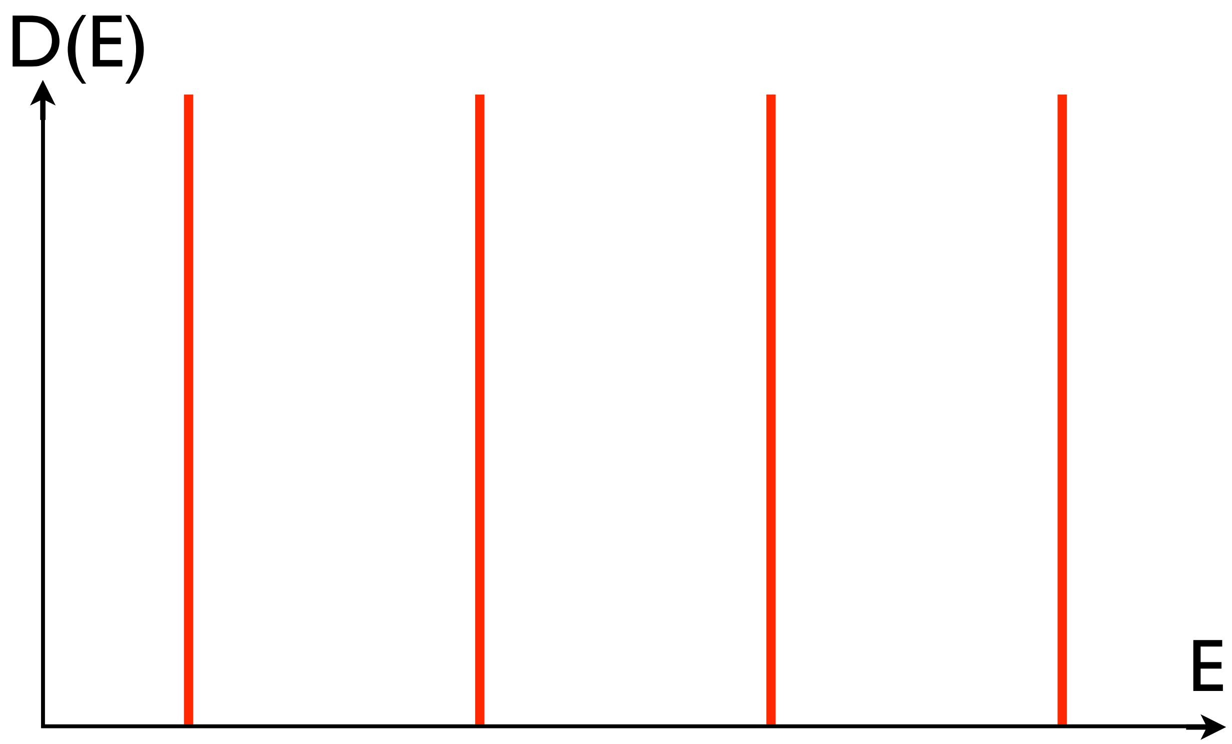
$$E = \frac{\hbar^2 k^2}{2m_e} \qquad k^2 = k_x^2 + k_y^2$$

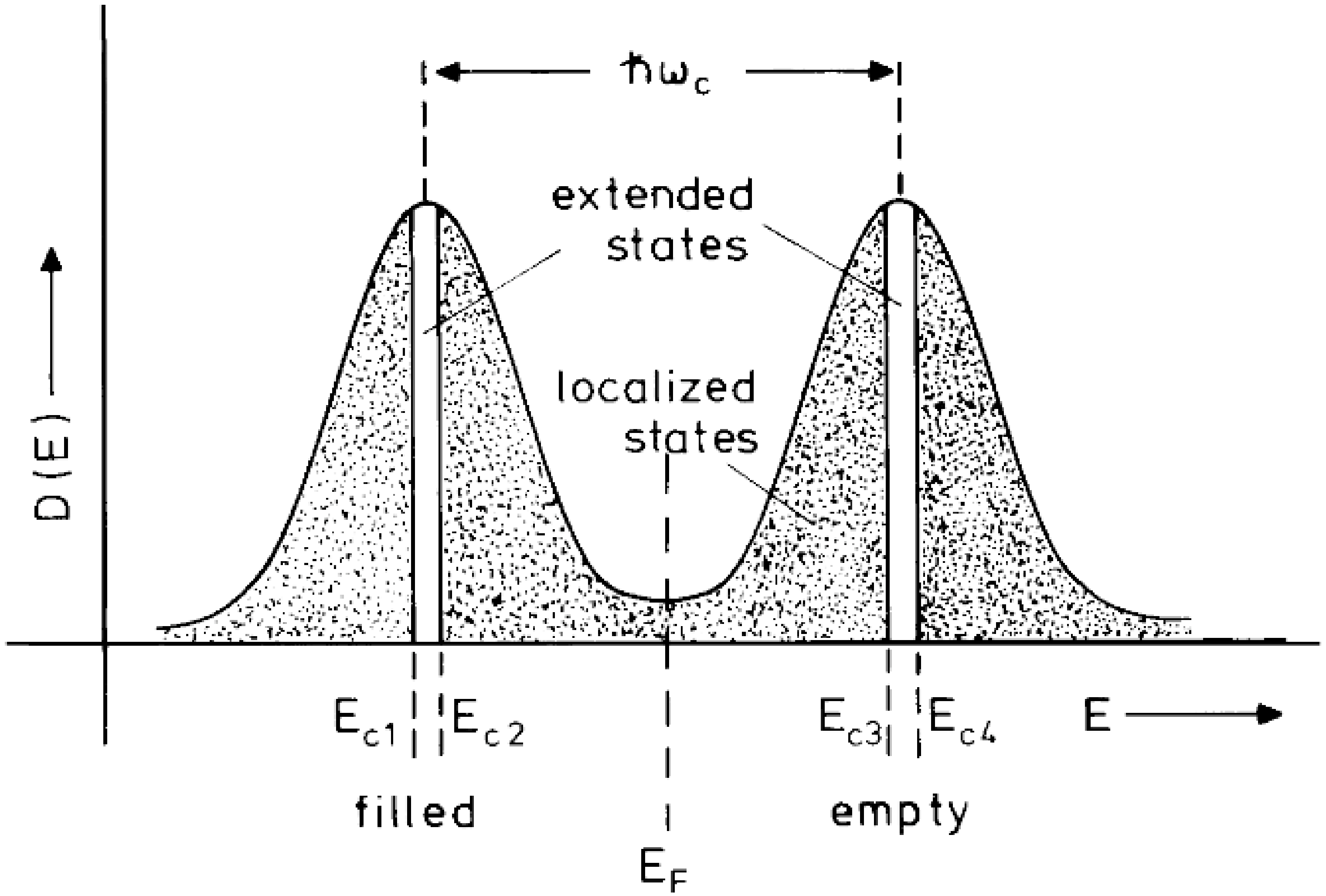
→ Discrete energy values with a high density of states

→ degeneracy N_S of the Landau levels

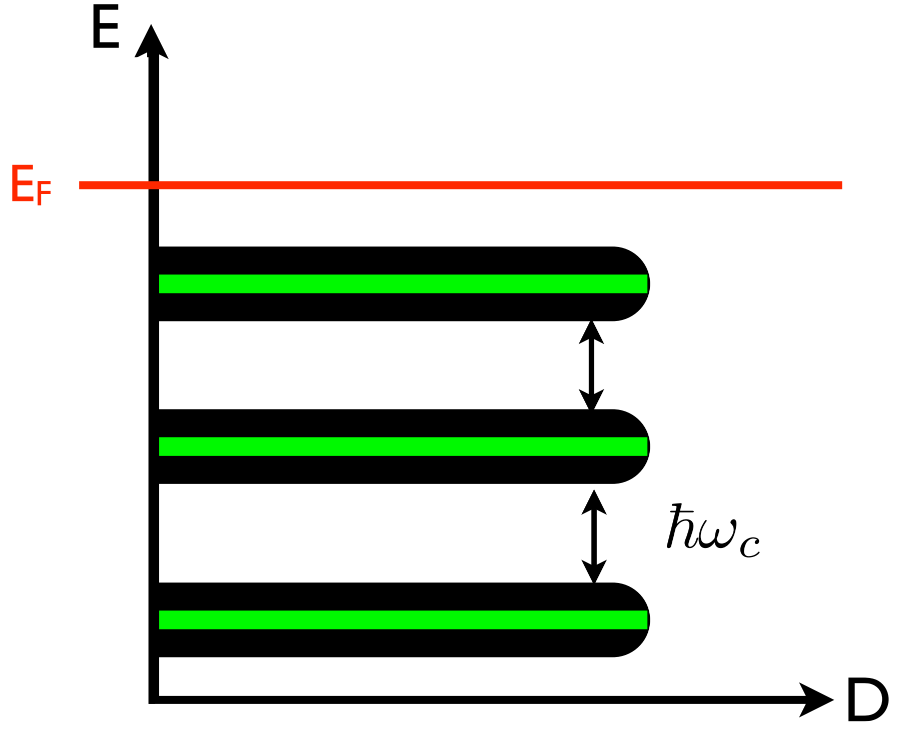
$$N_S = \frac{A}{2\pi l_B^2} = \frac{\Phi}{\Phi_0}$$

energy spectrum with and without disorder

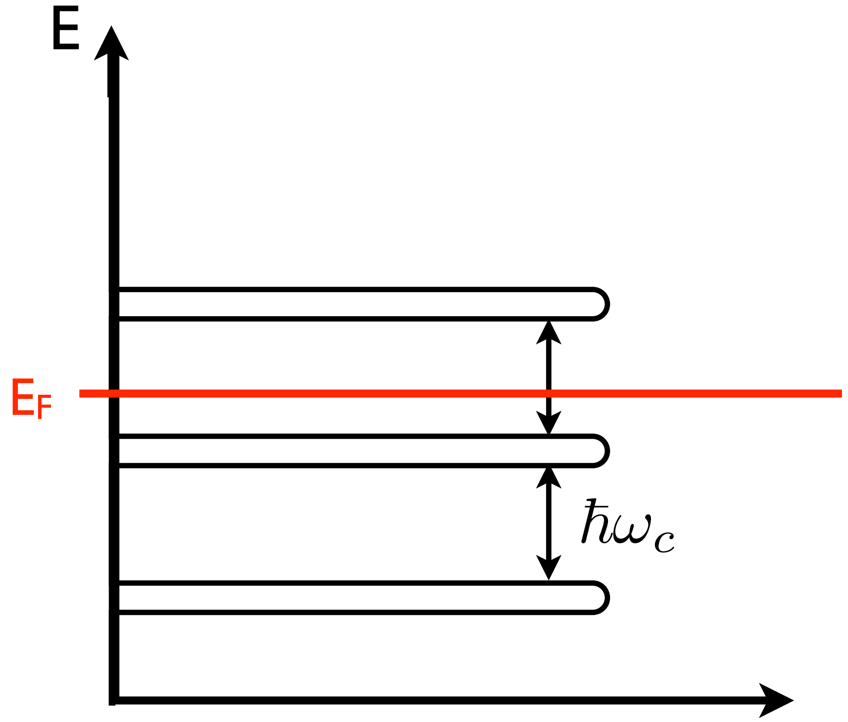


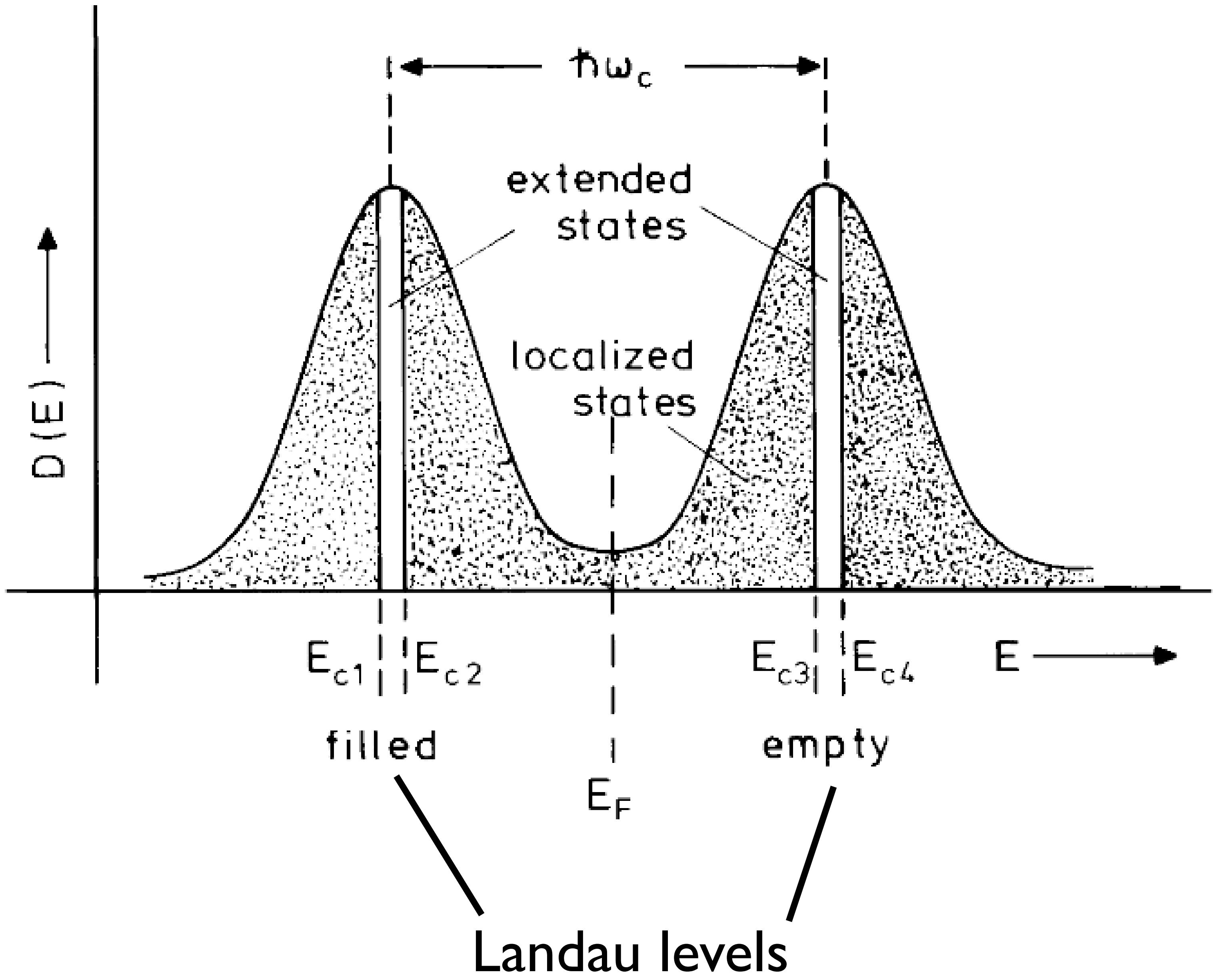


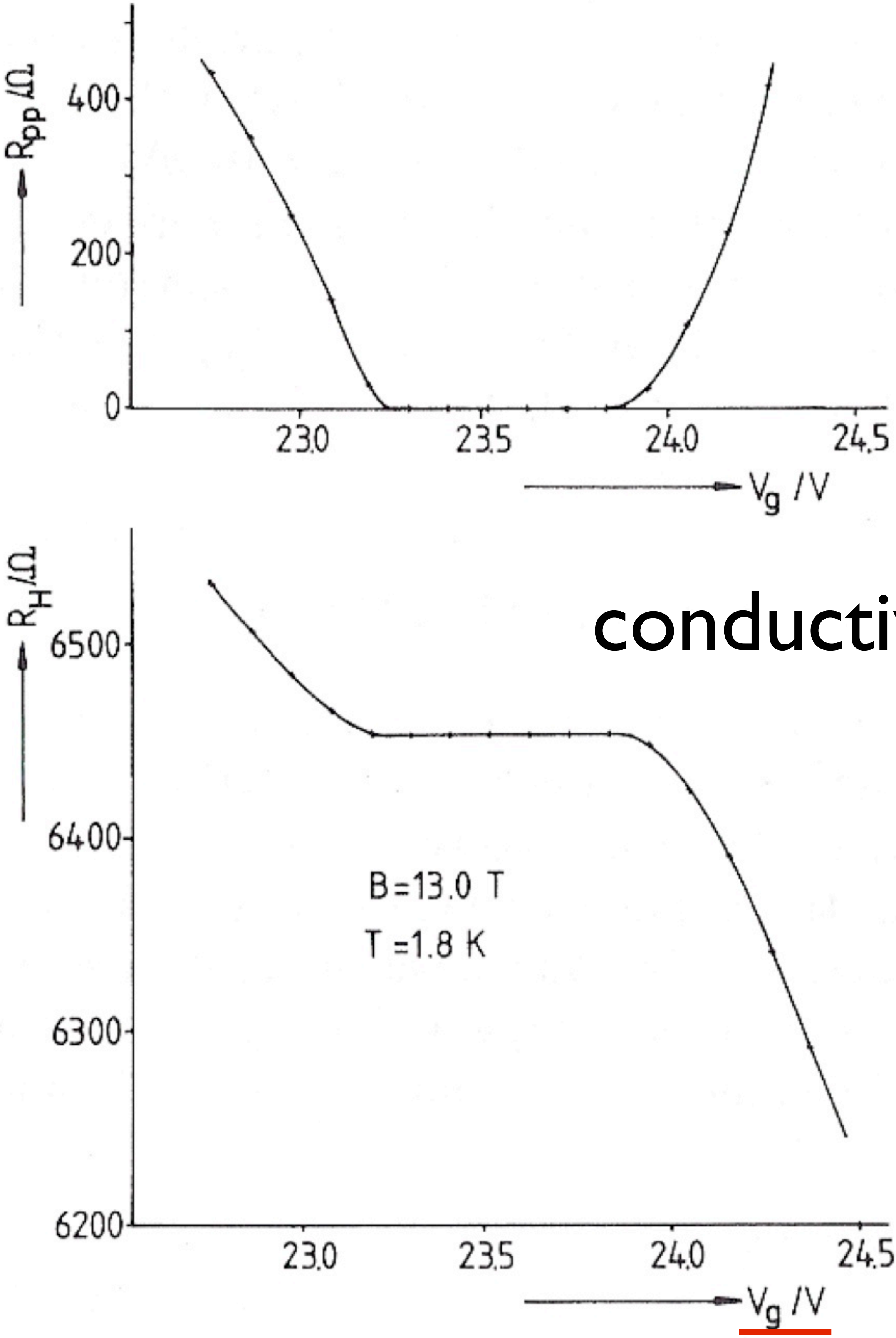
Increasing the magnetic field



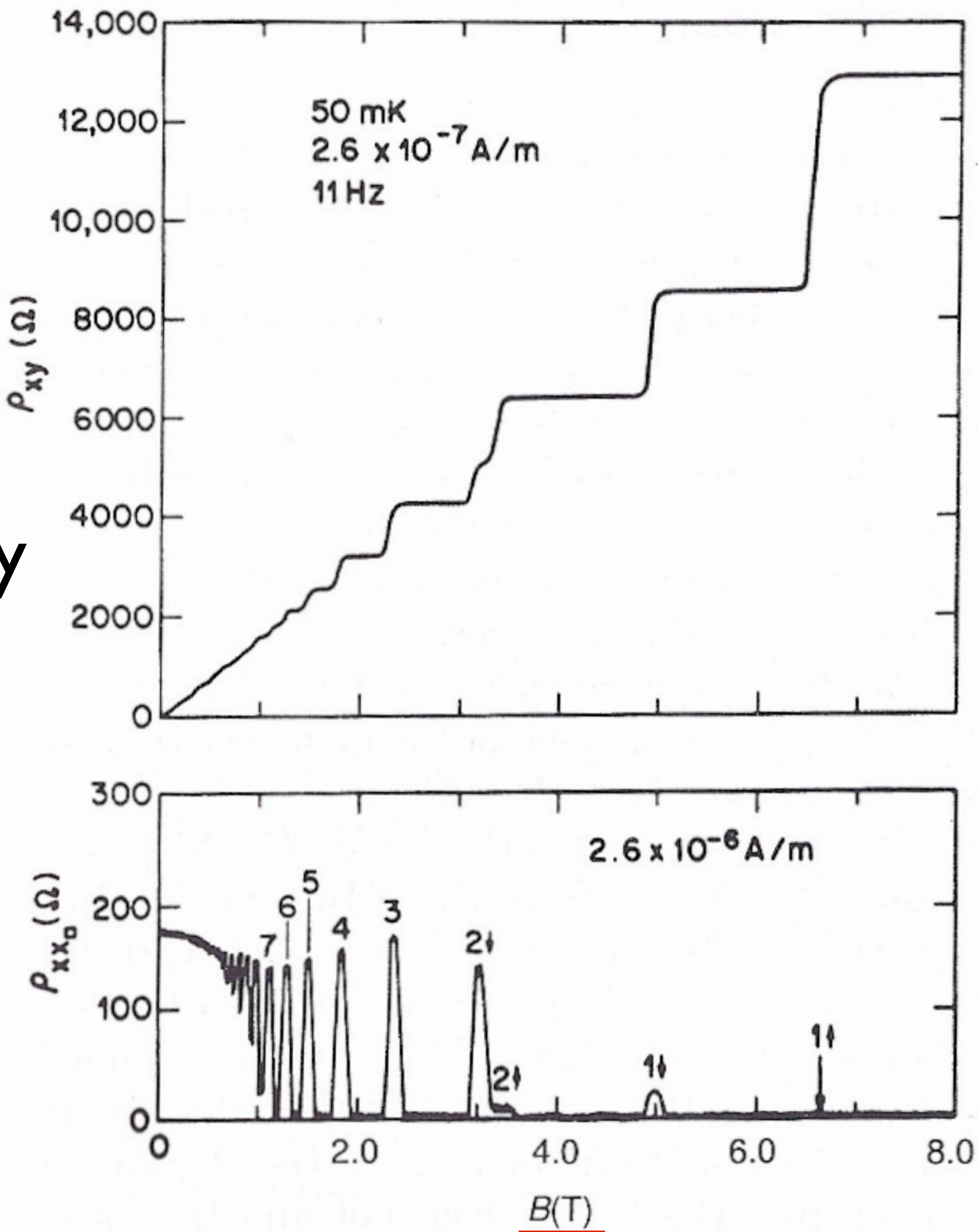
Increasing the densitiy of electrons







conductivity



Quantum Hall Effect in graphene

Here we have the same definitions as before:

$$\begin{aligned} \vec{A} &= (-By, 0) \\ l_B &= \sqrt{\frac{\hbar c}{eB}} \\ \omega_c &= \sqrt{2} \cdot \frac{v_F}{l_B} \end{aligned}$$

Dirac-equation for an electron moving in a 2-dimensional plane:

$$v_F \; \vec{\sigma} \cdot \vec{p} \; \psi(\vec{r}) \;=\; E\psi(\vec{r})$$

Switching on a magnetic field, the momentum operator has to be replaced:

$$\vec{p} \rightarrow \vec{p} - \frac{-e}{c}\vec{A}$$

$$-v_F \; \vec{\sigma} \cdot \left(i\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r}) \right) \psi(\vec{r}) \;=\; E\psi(\vec{r})$$

σ is a vector including the Pauli spin matrixes

$$\sigma_x \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \sigma_y \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

So the Dirac equation for this problem is written as follows

$$-v_F \left(\sigma_x \left(i\partial_x - \frac{e}{c}(-By) \right) + i\sigma_y \partial_y \right) \psi(\vec{r}) \;=\; E\psi(\vec{r})$$

The e^- move freely in x-direction, so we can separate the wave function in x- and y-direction:

$$\psi(\vec{r}) = \psi(x, y) = e^{ikx}\phi(y)$$

We obtain the Hamilton-operator for $\hat{H}\phi(y) = -\frac{E}{v_F}\phi(y)$ (1)

$$\hat{H} = \begin{pmatrix} 0 & \frac{eBy}{c} - k - \partial_y \\ \frac{eBy}{c} - k + \partial_y & 0 \end{pmatrix}$$

Multiply (1) with l_B and define a new variable $\boxed{\xi = \frac{y}{l_B} - l_B k}$

$$\begin{pmatrix} 0 & \xi + \partial_\xi \\ \xi - \partial_\xi & \end{pmatrix} \phi(\xi) = -\frac{\sqrt{2}\,E}{\omega_c}\phi(\xi)$$

Define again operators which should be ladder operators:

$$O = \frac{1}{\sqrt{2}}\left(\xi + \partial_\xi\right) \qquad O^+ = \frac{1}{\sqrt{2}}\left(\xi - \partial_\xi\right)$$

$$[O, O^+] = 1$$

$$\begin{pmatrix} 0 & O \\ O^+ & 0 \end{pmatrix} \phi(\xi) = -\frac{E}{\omega_c}\phi(\xi)$$

The wave function is a two component vector

$$\phi(\xi) = \begin{pmatrix} \phi_A(\xi) \\ \phi_B(\xi) \end{pmatrix}$$

where A and B describe the two sublattices in the hexagonal lattice.

Is it possible to write this Hamilton operator also that it looks like a harmonic oscillator?

→ Idea: Write the eigenvalue equation for H^2

$$\begin{pmatrix} 0 & O \\ O^+ & 0 \end{pmatrix}^2 \phi\ (\xi) = \frac{E^2}{\omega_c^2}\phi\ (\xi)$$

→ So we gain H^2, looking like a h.o.

$$\hat{H}^2 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O^+O\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\omega_c^2$$

O^+O is a number operator → we obtain the quantized eigenenergies for

sublattice A

$$E^2 = (\hbar\omega_c)^2 \cdot (m+1)$$

$$m = 0, 1, 2, ...$$

sublattice B

$$E^2 = (\hbar\omega_c)^2 \cdot n$$

$$n = 0, 1, 2, ...$$

The quantum numbers **n** and **m** are not independent:

$$\boxed{m \; = \; n - 1}$$

$$\rightarrow \qquad E_n \; = \; \hbar\omega_c\sqrt{n}$$

→ Every solution now can be constructed from the zero energy solution:

$$\phi_{N,\alpha}(\xi) \; = \; \begin{pmatrix} \psi_{N-1}(\xi) \\ \alpha \cdot \psi_N(\xi) \end{pmatrix}$$

To determine α, use $H\phi = E\phi$

$$\alpha = \mp 1$$

Zero energy state:

A solution with zero energy exists for n=0

$$\hat{H}\phi = \begin{pmatrix} 0 & O \\ O^{+} & 0 \end{pmatrix} \begin{pmatrix} \phi_A(\xi) \\ \phi_B(\xi) \end{pmatrix} = E \begin{pmatrix} \phi_A(\xi) \\ \phi_B(\xi) \end{pmatrix} = 0$$

$$\rightarrow \quad O\phi_B(\xi) = 0$$

$$O^{+}\phi_A(\xi) = 0$$

ground state

$$\Rightarrow \quad \phi_B(\xi) = \psi_{n=0}(\xi)$$

$$\Rightarrow \quad \phi_A(\xi) = 0$$

$$\phi_0(\xi) = \begin{pmatrix} 0 \\ \psi_{B,n=0}(\xi) \end{pmatrix}$$

ψ_N are the solutions of the harmonic oszillator

$$\psi_N = \frac{1}{\sqrt{2^N N!}} \; exp[-\frac{1}{2}\xi^2] \; H_N(\xi)$$

H_N is a Hermite polynomial

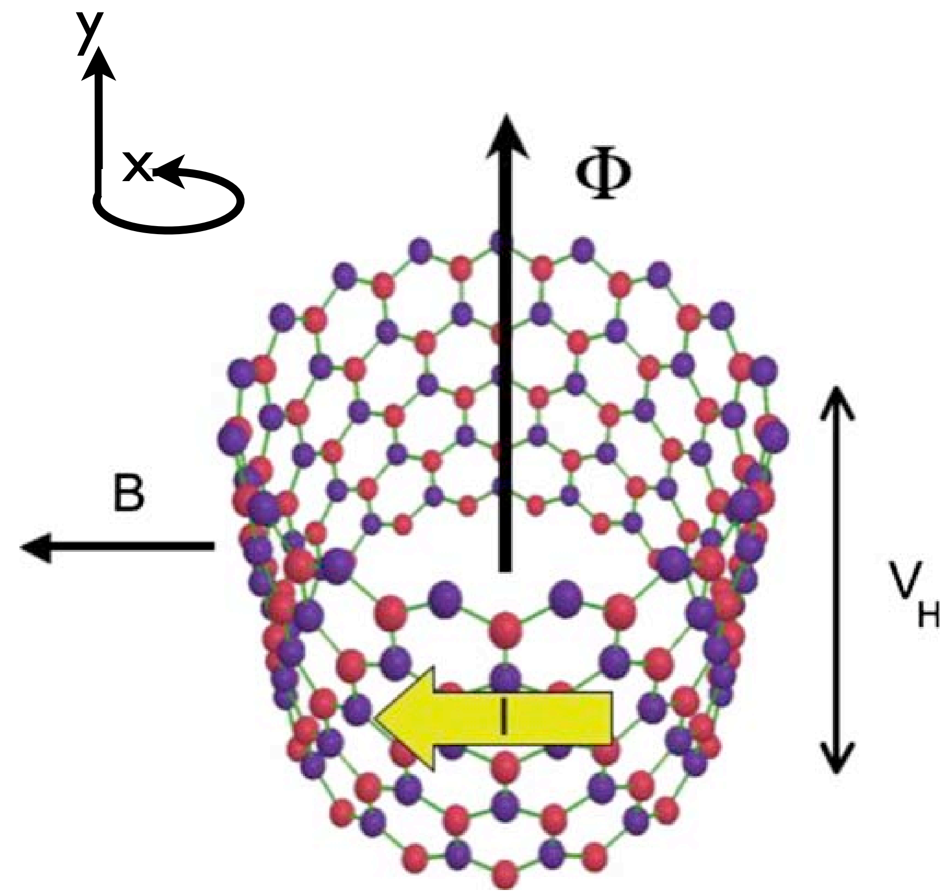
Remember the e^- wave function

$$\phi_{N,\mp}(\xi) = \begin{pmatrix} \psi_{N-1}(\xi) \\ \mp\psi_N(\xi) \end{pmatrix} \qquad E_\mp = \mp\hbar\omega_c\sqrt{N}$$

with its zero energy state for N=0

→ This state ist very important for understanding the QHE in graphene

The anomalous integer QHE



graphene ribbon, rolled up, current circling perpendicular to the magnetic field

→ current generates a magnetic flux Φ

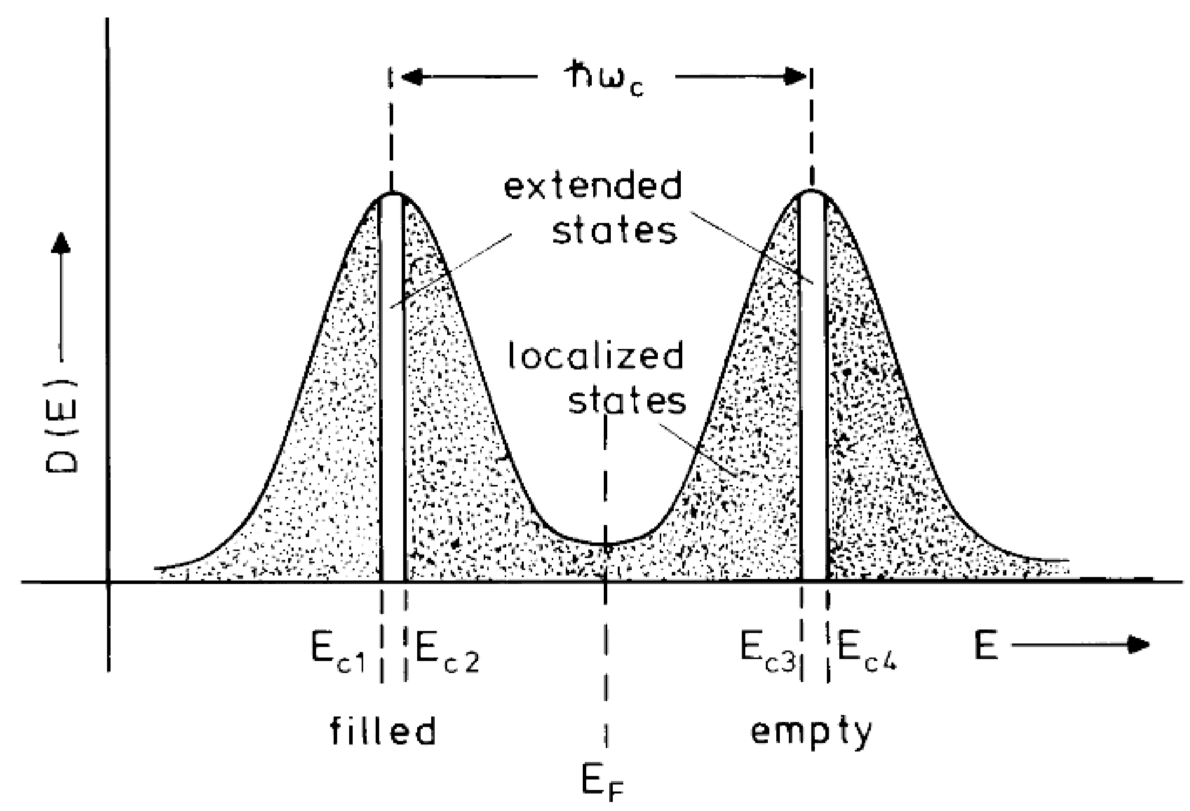
Changing the flux only would influence the extended states (they contribute to the current)

$$\phi(x + 2\pi R, y) = \phi(x, y)$$

$$\Delta\Phi = n \times \Phi_0 = n \times \frac{hc}{e}$$

Fermi energy in area of localized states
→ change of one flux quantum will not change number of occupied extended states (conductivity remains constant)

→ integer number of states leave the cylinder at one side, same number enters the cylinder on the other side when the flux changes



How many states cross the cylinder for $\delta\Phi = \Phi_0$?

Current is described by $$I = c\frac{\delta E}{\delta\Phi}$$ *E*: total energy in the system

Each Landau level contributes to the current with one state times its degeneracy g

→ g = 4 for graphene (two Dirac points times two spin states)

→ change of energy when flux is changed by one flux quantum:

$$\delta E = \pm 4N\underbrace{eV_H}_{\text{energy each electron has in y-direction}}$$

So the change in current in y-direction is given by:

$$\delta I = 4N\frac{e^2}{h}V_H$$

So we can calculate the Hall conductivity:

$$\sigma_{xy} = \frac{I}{V_H} = 4N\frac{e^2}{h}$$

This result is the same as for a 2DEG, but there would be only a factor 2 from the spin degeneracy in Landau levels.

For graphene there is a problem with this result!

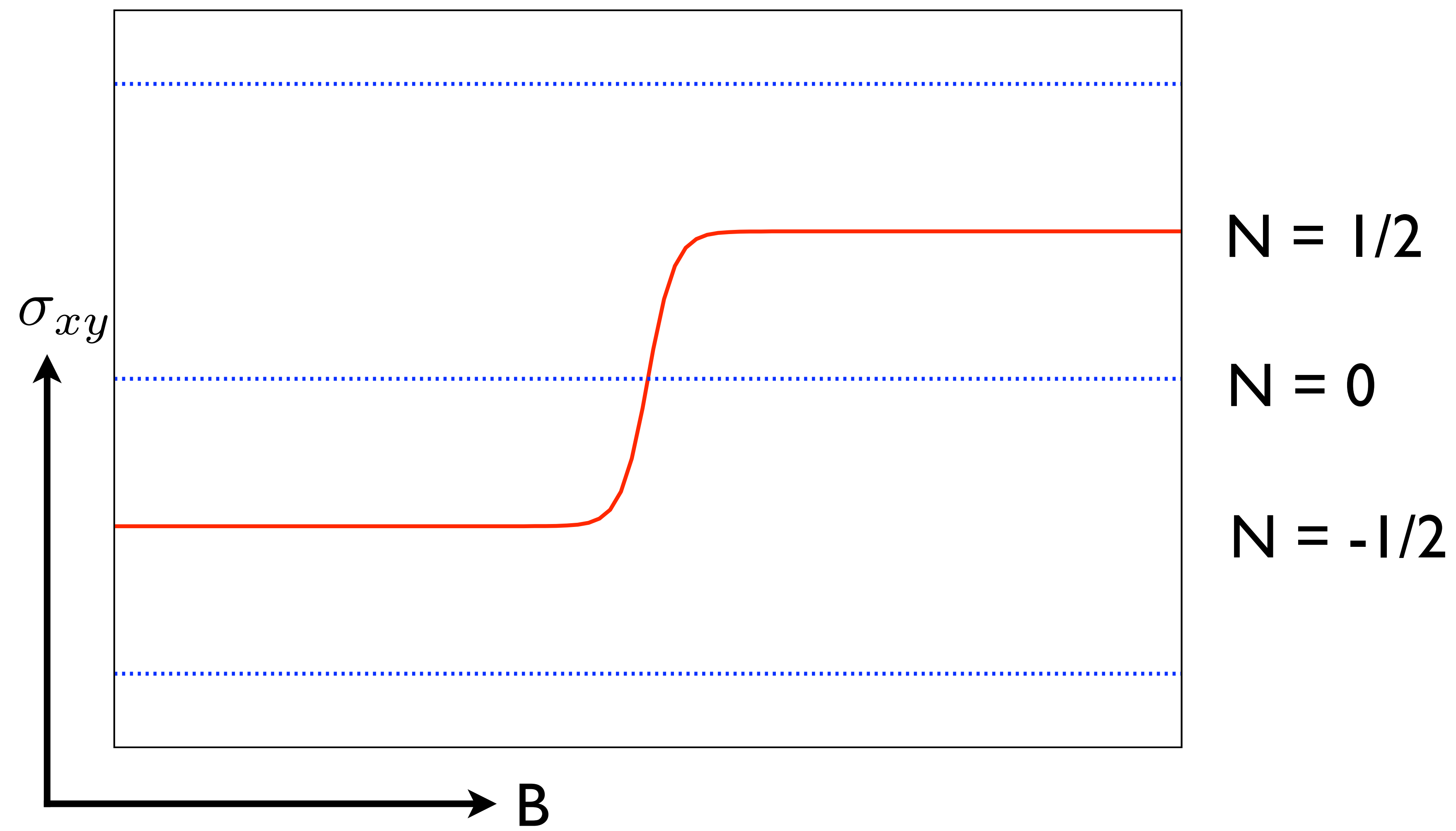
The equation above predicts a plateau of conductivity for N=0, which is not possible for graphene because there is a Landau level for N=0.

→ There is an area with extended states, so the conductivity changes by changing the flux.

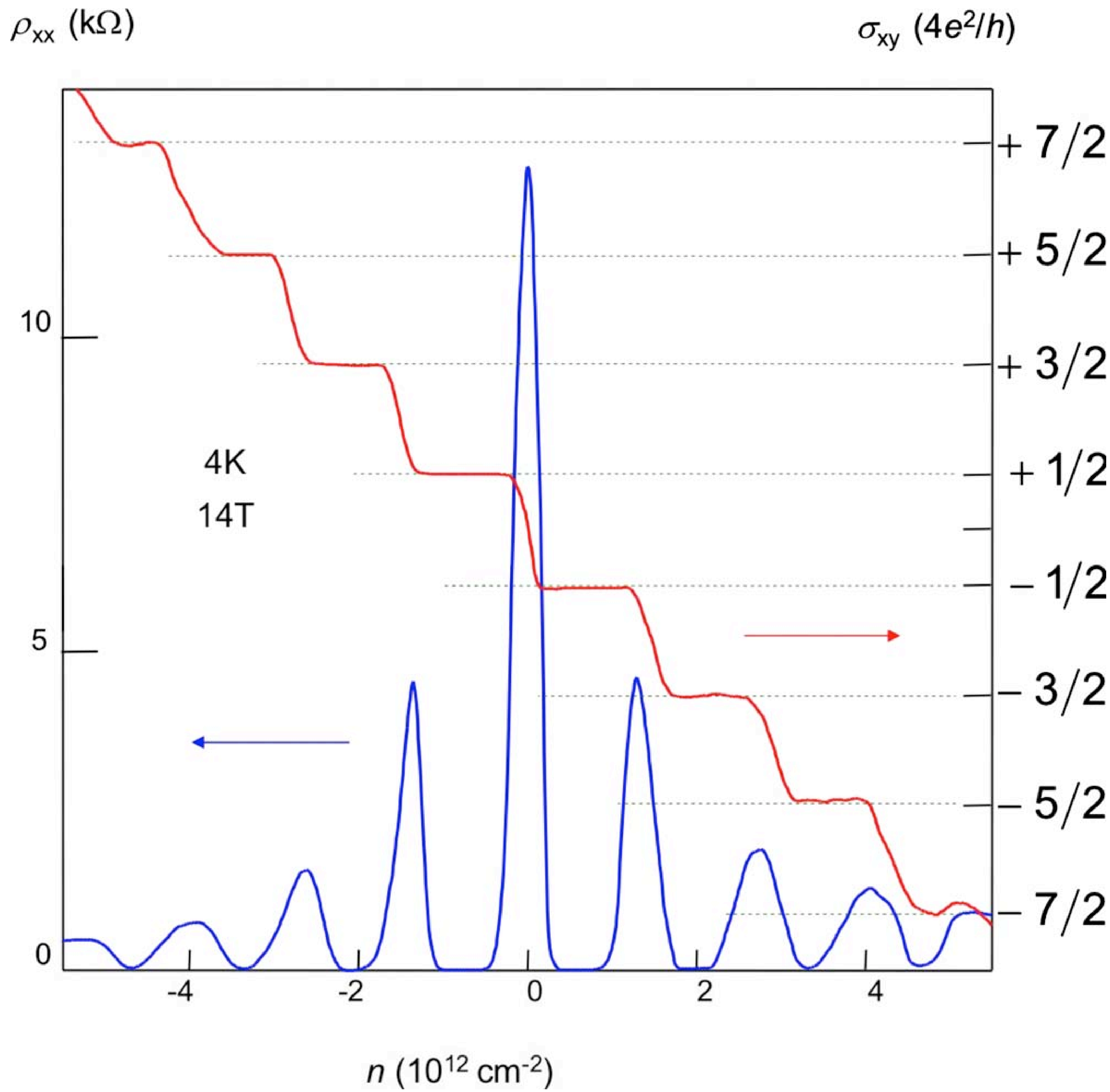
But: the lowest Landau level has some special properties, a different degeneracy

Lowest Landau level has half the degeneracy:

$$\sigma_{xy} = 4\frac{e^2}{h}(4N+2) = \frac{e^2}{h}\,2(2N+1)$$



This result was determined by experimental results, which is i.e. shown in the following picture, made by *Novoselov, Geim, Morozov, et al., 2005*



Summary

2DEG	graphene

- Schrödinger equation
- Landau levels E_n

$$E_n = \hbar\omega_c(n + \frac{1}{2})$$

- Hall-conductivity

$$\sigma_{xy} = \frac{e^2}{h}\ N$$

- Dirac equation
- Landau levels E_N

$$E_N = \mp\hbar\omega_c\sqrt{N}$$

- Hall-conductivity

$$\sigma_{xy} = \frac{e^2}{h}\ 2(2N+1)$$

References

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- K. von Klitzing, *The quantized hall effect, Reviews of Modern Physics 58, 519 (1986)*
- D. Yoshioka, *The Quantum Hall Effect, Springer, 2002*
- Ibach Lüth, *Festkörperphysik, Springer, 2008*